D-instantons and twistors: some exact results

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Abstract

We present some results on instanton corrections to the hypermultiplet moduli
space in Calabi–Yau compactifications of type-II string theories. Previously,
using twistor methods, only a class of D-instantons (D2-instantons wrapping
A-cycles) was incorporated exactly and the rest was treated only linearly. We
go beyond the linear approximation and give a set of holomorphic functions
which, through a known procedure, capture the effect of D-instantons at all
orders. Moreover, we show that for a sector where all instanton charges have
vanishing symplectic invariant scalar product, the hypermultiplet metric can be
computed explicitly.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Compactifications of type-II string theories on Calabi–Yau (CY) threefold yield $\mathcal{N} = 2$
supergravity coupled to vector multiplets (VM) and hypermultiplets (HM). At the two-
derivative level in the effective action, the moduli spaces of these two sectors are decoupled
and can be described independently. The vector multiplets are well known to be described by a
special Kähler geometry, which can be conveniently encoded into a holomorphic prepotential.
The hypermultiplet sector is much more complicated in two aspects. First, supersymmetry
restricts it to be described by the so-called quaternion–Kähler (QK) geometry [1]. Contrary to
the Kähler geometry, QK spaces are not described by any potential and thus it is difficult
to parametrize them in a simple way. Second, since the HM include the dilaton, the
effective action in the hypermultiplet sector receives corrections in the string coupling g_s,
both perturbative and non-perturbative. The non-perturbative corrections are related to D-
instantons arising as D-branes wrapping non-trivial cycles on the internal CY [2]. But due to
the lack of a well-established instanton calculus in string theory, the direct calculation of such
D-instanton corrections seems to be a very hard task. As a result, the exact metric on the HM
moduli space remains still unknown and represents a great challenge in string theory.

Recently, large progress in this direction has been achieved. A crucial step was provided by understanding how to overcome the first of the above-mentioned problems. Namely, it was realized that a suitable parametrization of QK spaces can be found using the so-called twistor techniques [3, 4], which in the physics literature appeared as the projective superspace method [5–9].

In particular, it was known for long time that $4d$-dimensional QK manifolds $\mathcal{M}$ are in one-to-one correspondence with $4d+$ four-dimensional hyper-Kähler (HK) cones, or Swann bundles $\mathcal{S}$ [4], which are hyper-Kähler manifolds with an additional homothetic Killing vector and an isometric $SU(2)$ action. The geometry of a HK manifold in turn can be encoded into the complex symplectic structure on its twistor space $\mathcal{Z}_\mathcal{S}$. It turns out that the latter is characterized by a set of holomorphic functions, which generate symplectomorphisms between local Darboux coordinates in various patches covering the twistor space [10]. These functions play the same role for the HK geometry as the holomorphic prepotential for the special Kähler geometry. However, the additional structure of the HK cone imposes some restrictions on these transition functions, which allow one to descend to the twistor space $\mathcal{Z}$ of the initial QK manifold [11]. As a result, the QK geometry can be obtained directly from the complex contact structure on $\mathcal{Z}$ [12] and is characterized by a related set of holomorphic functions interpreted now as complex contact transformations between different locally flat patches.

On the other hand, the absence of string instanton calculus was overcome by applying various non-perturbative (self)dualities of type-IIA and type-IIB string theories. In this way, applying $SL(2, \mathbb{Z})$ self-duality of type-IIB string theory to the perturbative HM metric found before in [13–15] (see also [16–20] for some earlier work), the authors of [21] were able to compute the D(-1) and D1-instanton corrections. Mirror symmetry further maps these corrections into D2-brane instantons of type-IIA theory compactified on the mirror CY threefold X [22]. However, this map recovers only instantons wrapping A-cycles in some symplectic polarization of $H_3(X)$. The contributions of other D2 instantons may then be restored using the symplectic invariance of type-IIA theory [23].

For these developments, the twistor and projective techniques mentioned above were indispensable since they allowed one to work with simple holomorphic functions encoding in a concise way complicated QK metrics. There is however an important difference in the twistor description of A-type D2 instantons and more general ones. Although the transition functions on the twistor space $\mathcal{Z}$ define the metric on the associated QK space $\mathcal{M}$ in a unique way, this requires knowledge of the so-called contact twistor lines, representing local Darboux coordinates on $\mathcal{Z}$ in terms of coordinates on $\mathcal{M}$ and the coordinate $\mathbf{z}$ on the $\mathbb{C}P^1$ fiber. Generically, this requires a solution of complicated integral equations and cannot be done explicitly. But in the special case when the QK space has $d + 1$ commuting isometries, the solution is known and can be described by the so-called $\mathcal{O}(2)$ multiplets. This is precisely the case of the HM moduli space with only A-type D2 instantons included. All other instantons break the isometries to discrete subgroups and a solution for the twistor lines can be written only as a perturbative series in instantons. In particular, the linear approximation was explicitly found in [23].

The aim of this paper is to present some *exact* results, going beyond the linear approximation. The paper consists of two parts. The first part concerns the underlying mathematical construction, whereas the second addresses the physical problem of the HM moduli space.

The first part gives mostly a very brief review of the previous works [10, 11], introducing the necessary notions and notations. It is split into two parts dealing with HK and QK cases, respectively. Besides the simple review, it contains also some new results. In particular, we derive the formula (2.16) for the exact 'Lagrangian', related to the Kähler potential on

HK space by the Legendre transform, and establish a relation with the projected superspace formulation of [9].

In the second part we elaborate on the results of [23], where the contributions of all D2-instantons were found in the linear approximation. First, we show that with a suitable choice of transition functions, favored by symplectic invariance, in the case of a single D2-instanton the linear approximation becomes exact. It is not something unexpected because by a suitable symplectic transformation any D2-instanton can be mapped to a D2-brane wrapping an A-cycle, for which the description in terms of $\mathcal{O}(2)$ multiplets is applied and the contact twistor lines are known exactly. Thus, our result can be considered as a non-trivial test satisfied by the proposed transition functions.

Then the inclusion of several charges is considered and, although we are not able to compute explicitly the twistor lines, we present some new insights. In particular, we give a full consistent set of transition functions incorporating all D-instantons, improving a somewhat naive proposal of [23]. This set is necessary for the complete description of the HM moduli space and its twistor space. Furthermore, we find that a certain sector, consisting of D2-instantons with all 'mutually local charges', admits the exact description similar to that of the single-charge case. Finally, we discuss the consistency of our results with symplectic invariance and the so-called wall-crossing conditions [24, 25].

2. Twistor description of HK and QK spaces

2.1. Kähler potential for HK manifolds

We start from the twistor approach to description of HK spaces. Although it is not directly needed for our discussion of the HM moduli space in the following sections, we include it to present an explicit formula for the Kähler potential in terms of twistor lines. Our discussion closely follows [10].

The twistor space is a $\mathbb{C}P^1$ bundle over the initial HK manifold. It is equipped with a complex structure and a holomorphic two-form:

$$\Omega(\zeta) = \omega^+ - \mathrm{i}\zeta\omega^3 + \zeta^2\omega^-, \tag{2.1}$$

where ζ is a coordinate on $\mathbb{C}P^1$ and ω^i are Kähler forms associated with three complex structures J^i carried by any HK manifold. Note that ω^+ (ω^-) is (anti-)holomorphic with respect to the complex structure J^3.

In fact, Ω is a section of a two-form-valued $\mathcal{O}(2)$ bundle on $\mathbb{C}P^1$ [5] which is reflected in the fact that the representation (2.1) diverges at $\zeta = \infty$. This signifies that one should cover $\mathbb{C}P^1$ by two open intersecting patches, $\mathcal{U}_0$ and $\mathcal{U}_\infty$, around the north and south poles, respectively, and the holomorphic two-form is represented in every patch as

$$\Omega^{[0]}(\zeta) = \Omega(\zeta), \qquad \Omega^{[\infty]}(\zeta) = f_{0\infty}^{-2}\Omega(\zeta) = \omega^- - \mathrm{i}\zeta^{-1}\omega^3 + \zeta^{-2}\omega^+, \tag{2.2}$$

where $f_{0\infty}^2 = \zeta^2$ is the transition function of the line bundle $\mathcal{O}(2)$ over $\mathbb{C}P^1$. More generally, we have to work with a set of patches $\mathcal{U}_i$ and representatives $\Omega^{[i]}$ such that on every overlap $\mathcal{U}_i \cap \mathcal{U}_j$ they are related by

$$\Omega^{[i]} = f_{ij}^2\Omega^{[j]} \quad \mathrm{mod}\, \mathrm{d}\zeta. \tag{2.3}$$

In every patch one can introduce a local system of Darboux coordinates

$$\Omega^{[i]} = \mathrm{d}\mu_I^{[i]} \wedge \mathrm{d}\nu_{[i]}^I. \tag{2.4}$$

Different systems on the overlaps of two patches are related by symplectomorphisms, which can be expressed through generating functions $S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta)$ of the initial 'position' and final 'momentum' coordinates. Then the gluing conditions take the following form

$$\nu^I_{[j]} = \partial_{\mu^{[j]}_I} S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta), \qquad \mu^{[i]}_I = f^2_{ij} \partial_{\nu^I_{[i]}} S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta). \qquad (2.5)$$

The transition functions $S^{[ij]}$ contain all information about the twistor space and the underlying HK manifold. They can be chosen arbitrarily up to some consistency and reality conditions, and define the HK space uniquely up to some gauge freedom [10]. All geometric information can be restored once we solved the gluing conditions (2.5) for the coordinates ν^I and μ_I as functions of ζ. Such functions are called twistor lines. The free parameters of the solution play the role of coordinates on the HK base.

Here we will consider the situation where ν^I and μ_I are perturbations of global sections of $\mathcal{O}(2)$ and $\mathcal{O}(0)$, respectively. Then it is convenient to redefine

$$\eta^I_{[i]}(\zeta) \equiv \zeta^{-1} f^2_{0i} \nu^I_{[i]}(\zeta) \qquad (2.6)$$

and choose the generating functions as

$$S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta) = \zeta f^{-2}_{0j} \left(\eta^I_{[i]} \mu^{[j]}_I - H^{[ij]}(\eta_{[i]}, \mu^{[j]}, \zeta) \right). \qquad (2.7)$$

However, $H^{[ij]}$ are not assumed to be infinitesimal and therefore we do not actually impose any restrictions. The gluing conditions then become

$$\eta^I_{[j]} = \eta^I_{[i]} - \partial_{\mu^{[j]}_I} H^{[ij]}, \qquad \mu^{[j]}_I = \mu^{[i]}_I + \partial_{\eta^I_{[i]}} H^{[ij]}. \qquad (2.8)$$

These conditions can be rewritten as integral equations, which are suitable for the perturbative treatment (see appendix A.2)

$$\eta^I_{[i]} = \eta^I_{(0)} + \hat{\eta}^I_{[i]}, \qquad \hat{\eta}^I_{[i]} = -\frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i}\zeta'} \frac{\zeta'^3 + \zeta^3}{\zeta \zeta'(\zeta' - \zeta)} H^{[ij]I}(\zeta'),$$

$$\mu^{[i]}_I = \frac{\mathrm{i}}{2}\varrho_I + \hat{\mu}^{[i]}_I, \qquad \hat{\mu}^{[i]}_I = \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i}\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} H^{[ij]}_I(\zeta'), \qquad (2.9)$$

where the variable ζ is inside the contour C_i surrounding $\mathcal{U}_i$ in the counterclockwise direction, $H^{[ij]I} \equiv \partial_{\mu^{[j]}_I} H^{[ij]}$, $H^{[ij]}_I \equiv \partial_{\eta^I_{[i]}} H^{[ij]}$ and

$$\eta^I_{(0)}(\zeta) = \frac{\nu^I}{\zeta} + x^I - \bar{\nu}^I \zeta. \qquad (2.10)$$

Of course, the integral equations (2.9) are as difficult as the initial equations (2.8). But on the other hand, they introduce explicitly the coordinates on the HK base, ν^I, $\bar{\nu}^I$, x^I and ϱ_I, and allow one to get the complex structure together with the Kähler potential. Indeed, due to (2.1), the holomorphic form ω^+ is obtained as a constant term in the small ζ-expansion of (2.4) in the patch $\mathcal{U}_0$:

$$\omega^+ = \mathrm{d}\left(\frac{\mathrm{i}}{2}\varrho_I + \frac{1}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} H_I \right) \wedge \mathrm{d}\left(\nu^I - \frac{1}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} H^I \right), \qquad (2.11)$$

where, to avoid cluttering, we omitted the sum over different patches. From this result, we read the complex coordinates on our HK space

$$u^I \equiv \nu^I - \frac{1}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} H^I, \qquad w_I \equiv \frac{\mathrm{i}}{2}\varrho_I + \frac{1}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} H_I. \qquad (2.12)$$

Similarly one can get a formula for the Kähler form ω^3. It imposes the following conditions on the Kähler potential (the index on K denotes the derivative w.r.t. the corresponding variable)

$$K_{u^I} = \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta^2} H_I, \qquad K_{w_I} = -x^I + \frac{1}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} H^I. \qquad (2.13)$$

Let us trade the Kähler potential for its Legendre transform

$$K(u, \bar{u}, w, \bar{w}) = \langle \mathcal{L}(u, \bar{u}, x, \varrho) - x^I(w_I + \bar{w}_I) \rangle_{x^I}. \tag{2.14}$$

In the projective approach the function $\mathcal{L}$ appears as a Lagrangian on the projective superspace [5, 7, 9]. From (2.13), the Lagrangian must satisfy

$$\mathcal{L}_{u^I} = \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta^2} H_I, \qquad \mathcal{L}_{x^I} = \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} H_I, \qquad \mathcal{L}_{\varrho_I} = \frac{\mathrm{i}}{2} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} H^I. \tag{2.15}$$

Remarkably, these equations are integrable (see appendix A.1) and solved by

$$\mathcal{L} = \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} (H - \hat{\mu}_I \partial_{\mu_I} H). \tag{2.16}$$

Together with (2.14), this result gives an explicit representation for the Kähler potential.

Our result can also be rewritten in the form found in [9]. Using (2.8) and (2.12), it is easy to check that

$$K = -\oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \tilde{S} - w_I \oint_{C_0} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \eta^I_{[0]} + \bar{w}_I \oint_{C_\infty} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \eta^I_{[\infty]}, \tag{2.17}$$

where

$$\tilde{S}^{[ij]}(\eta_{[i]}, \eta_{[j]}) = \zeta^{-1} f_{0j}^2 \langle S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta) - \nu^I_{[j]} \mu^{[j]}_I \rangle_{\mu^{[j]}_I} \tag{2.18}$$

are generating functions of symplectomorphisms written as functions of two 'positions'. This representation coincides with equation (4.1) of [9], where the case of only two patches was considered, provided $\zeta \eta^I_{[0]}$ and $-\zeta^{-1} \eta^I_{[\infty]}$ are identified as arctic and antarctic multiplets, respectively.

As usual, the metric can be computed without knowing x^I as a function of $u^I, \bar{u}^I, w_I, \bar{w}_I$ explicitly. It is expressed through the derivatives of the Lagrangian as

$$\begin{aligned}
K_{u^I \bar{u}^J} &= \mathcal{L}_{u^I \bar{u}^J} - \mathcal{L}_{u^I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L \bar{u}^J}, \\
K_{u^I \bar{w}_J} &= \mathcal{L}_{u^I x^K} \mathcal{L}^{x^K x^J} + \mathrm{i}[\mathcal{L}_{u^I \varrho_J} - \mathcal{L}_{u^I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L \varrho_J}], \\
K_{w_I \bar{u}^J} &= \mathcal{L}^{x^I x^K} \mathcal{L}_{x^K \bar{u}^J} - \mathrm{i}[\mathcal{L}_{\varrho_I \bar{u}^J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L \bar{u}^J}], \\
K_{w_I \bar{w}_J} &= \mathcal{L}_{\varrho_I \varrho_J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L \varrho_J} - \mathcal{L}^{x^I x^J} + \mathrm{i}[\mathcal{L}^{x^I x^K} \mathcal{L}_{x^K \varrho_J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^J}],
\end{aligned} \tag{2.19}$$

where $\mathcal{L}^{x^I x^J}$ denotes the inverse of the matrix $\mathcal{L}_{x^I x^J}$. In appendix A.1 we also provide formulae for the inverse metric.

2.2. Twistor space of QK spaces

As was mentioned in section 1, QK spaces are in one-to-one correspondence with HK cones and thus can be described by the formalism of the previous subsection, provided one imposes additional constraints taking into account that the HK space in question is a cone. In particular, the transition functions $H^{[ij]}$ must be homogeneous of first degree in the arguments $\eta^I_{[i]}$ and independent of ζ [11, 26, 27]. On the other hand, in [11, 12] it was shown that QK spaces can be described directly in terms of their twistor spaces $\mathcal{Z}$ and all geometric information is again contained in transition functions relating different sets of coordinates analogous to the Darboux coordinates from the previous subsection. In this case, these are canonical coordinates for the contact one-form

$$\mathcal{X}^{[i]} \equiv \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]} \mathrm{d}\tilde{\xi}^{[i]}_\Lambda. \tag{2.20}$$

They can be very easily related to the coordinates ν^I, μ_I used above and the $\mathcal{O}(2)$-valued complex Liouville form on $\mathcal{Z}_\mathcal{S}$:

$$\xi^\Lambda_{[i]} = \nu^\Lambda_{[i]}/\nu^\alpha_{[i]}, \qquad \tilde{\xi}^{[i]}_\Lambda = \mu^{[i]}_\Lambda, \qquad \alpha^{[i]} = \mu^{[i]}_\alpha, \qquad \nu^\alpha_{[i]}\mathcal{X}^{[i]} = \nu^I_{[i]}\,\mathrm{d}\mu^{[i]}_I, \qquad (2.21)$$

where we have singled out one coordinate $\nu^\alpha_{[i]}$, and denoted by $\nu^\Lambda_{[i]}$ the remaining d coordinates. One can show that the HK cone conditions ensure that $\xi^\Lambda_{[i]}$, $\tilde{\xi}^{[i]}_\Lambda$ and $\alpha^{[i]}$ are all functions of the coordinates x^μ on $\mathcal{M}$ and the coordinate $\mathbf{z}$ parametrizing the $\mathbb{C}P^1$ fiber of $\mathcal{Z}$.

On the overlap of two patches, the contact form satisfies [11]

$$\mathcal{X}^{[i]} = \hat{f}^2_{ij}\mathcal{X}^{[j]}, \qquad \hat{f}^2_{ij} \equiv f^2_{ij}\nu^\flat_{[j]}/\nu^\flat_{[i]} = \eta^\flat_{[j]}/\eta^\flat_{[i]}, \qquad (2.22)$$

whereas the canonical coordinates are related by contact transformations

$$\xi^\Lambda_{[j]} = \xi^\Lambda_{[i]} + T^\Lambda_{[ij]}, \qquad \tilde{\xi}^{[j]}_\Lambda = \tilde{\xi}^{[i]}_\Lambda + \tilde{T}^{[ij]}_\Lambda, \qquad \alpha^{[j]} = \alpha^{[i]} + \tilde{T}^{[ij]}_\alpha, \qquad (2.23)$$

where we abbreviated

$$\begin{aligned}
T^\Lambda_{[ij]} &\equiv -\partial_{\tilde{\xi}^{[j]}_\Lambda}H^{[ij]} + \xi^\Lambda_{[j]}\partial_{\alpha^{[j]}}H^{[ij]}, \\
\tilde{T}^{[ij]}_\Lambda &\equiv \partial_{\xi^\Lambda_{[i]}}H^{[ij]}, \qquad \tilde{T}^{[ij]}_\alpha \equiv H^{[ij]} - \xi^\Lambda_{[i]}\partial_{\xi^\Lambda_{[i]}}H^{[ij]}
\end{aligned} \qquad (2.24)$$

with $H^{[ij]}$ being a general function of $\xi^\Lambda_{[i]}$, $\tilde{\xi}^{[j]}_\Lambda$ and $\alpha^{[j]}$. Moreover, the coefficients $\hat{f}^2_{ij}$, appearing in (2.22), are also determined by $H^{[ij]}$,

$$\hat{f}^2_{ij} = 1 - \partial_{\alpha^{[j]}}H^{[ij]}. \qquad (2.25)$$

To deal with the gluing conditions, we assume that the section $\nu^\flat$ on the twistor space $\mathcal{Z}_\mathcal{S}$ of the Swann bundle has only two zeros situated at the centers of patches $\mathcal{U}_\pm$. (This assumption is valid in the important particular case when $\nu^\flat$ is a global $\mathcal{O}(2)$ section.) By an SU(2) rotation one can always bring them to the points $\mathbf{z} = 0$ and $\mathbf{z} = \infty$, respectively. Then these points correspond to simple poles of $\xi^\Lambda_{[\pm]}$. These are the only singularities of the contact twistor lines $\xi^\Lambda_{[i]}$, $\tilde{\xi}^{[i]}_\Lambda$, $\alpha^{[i]}$ except those introduced by the so-called 'anomalous dimensions' c_Λ, c_α. The latter originate from possible logarithmic branch cuts on $\mathcal{Z}_\mathcal{S}$ and represent additional input supplementing transition functions [11]. As a result, the gluing conditions (2.23) for the contact twistor lines can be written as the following integral equations

$$\xi^\Lambda_{[i]}(\mathbf{z}, x^\mu) = A^\Lambda + \mathbf{z}^{-1}Y^\Lambda - \mathbf{z}\bar{Y}^\Lambda + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi\mathrm{i}\mathbf{z}'}\frac{\mathbf{z}'+\mathbf{z}}{\mathbf{z}'-\mathbf{z}}T^\Lambda_{[+j]}(\mathbf{z}'),$$

$$\tilde{\xi}^{[i]}_\Lambda(\mathbf{z}, x^\mu) = \frac{\mathrm{i}}{2}B_\Lambda + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi\mathrm{i}\mathbf{z}'}\frac{\mathbf{z}'+\mathbf{z}}{\mathbf{z}'-\mathbf{z}}\tilde{T}^{[+j]}_\Lambda(\mathbf{z}') + c_\Lambda\log\mathbf{z}, \qquad (2.26)$$

$$\alpha^{[i]}(\mathbf{z}, x^\mu) = \frac{\mathrm{i}}{2}B_\alpha + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi\mathrm{i}\mathbf{z}'}\frac{\mathbf{z}'+\mathbf{z}}{\mathbf{z}'-\mathbf{z}}\tilde{T}^{[+j]}_\alpha(\mathbf{z}') + c_\alpha\log\mathbf{z} + c_\Lambda(Y^\Lambda\mathbf{z}^{-1} + \bar{Y}^\Lambda\mathbf{z}),$$

where Y^Λ, A^Λ, B_Λ, B_α are free parameters playing the role of coordinates on $\mathcal{M}$. They contain one parameter more than the dimension of $\mathcal{M}$ because the overall phase rotation of Y^Λ can be absorbed by a redefinition of the $\mathbb{C}P^1$ coordinate $\mathbf{z}$.

Similarly to the holomorphic two-form on $\mathcal{Z}_\mathcal{S}$ (2.1), the contact form on $\mathcal{Z}$ is restricted to have the following expansion

$$\mathcal{X}^{[i]} = 2\frac{e^{\Phi_{[i]}}}{\mathbf{z}}(\mathrm{d}\mathbf{z} + p_+ - \mathrm{i}p_3\mathbf{z} + p_-\mathbf{z}^2), \qquad (2.27)$$

where $\vec{p}$ is the $SU(2)$ part of the Levi–Civita connection on $\mathcal{M}$ and the function $\Phi_{[i]}$ is the so-called 'contact potential'. In general it is defined only locally and therefore carries the

index of the patch $\mathcal{U}_i$. This is an important object since its real part provides a Kähler potential for the Kähler–Einstein metric on $\mathcal{Z}$

$$K_{\mathcal{Z}}^{[i]} = \log \frac{1 + \mathbf{z}\bar{\mathbf{z}}}{|\mathbf{z}|} + \text{Re}\,\Phi_{[i]}(x^\mu, \mathbf{z}). \tag{2.28}$$

The gluing conditions for the contact potential are determined by (2.25)

$$\Phi_{[i]} - \Phi_{[j]} = \log(1 - \partial_{\alpha^{[j]}} H^{[ij]}) \tag{2.29}$$

and the potential must be regular everywhere. The condition (2.29) can be solved in terms of contact twistor lines as

$$\Phi_{[i]} = \phi - \frac{1}{2} \sum_j \oint_{C_j} \frac{d\mathbf{z}'}{2\pi i \mathbf{z}'} \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \log(1 - \partial_{\alpha^{[j]}} H^{[+j]}(\mathbf{z}')). \tag{2.30}$$

The first term in (2.30), which is a real constant, can actually be computed explicitly and is given by

$$e^\phi = \frac{1}{4} \sum_j \oint_{C_j} \frac{d\mathbf{z}'}{2\pi i \mathbf{z}'} (\mathbf{z}'^{-1} Y^\Lambda - \mathbf{z}' \bar{Y}^\Lambda) \tilde{T}_\Lambda^{[+j]}(\mathbf{z}') + \frac{1}{2}(c_\Lambda A^\Lambda + c_\alpha). \tag{2.31}$$

Although this result was obtained in [11] only in the first order in perturbations around the case with $d+1$ commuting isometries (the non-perturbed transition functions $H^{[ij]}$ are independent of $\tilde{\xi}_\Lambda$ and α), it is easy to see that it holds for all orders. The contact twistor lines and the contact potential (2.30) provide sufficient information to compute the metric on $\mathcal{M}$. A procedure to do this is described in detail in [11, 23].

Note that in the important particular case where the transition functions are independent of α, the above description crucially simplifies: $\hat{f}_{ij}^2 = 1$ and the contact potential is globally defined coinciding with its constant part ϕ. This is the case for the hypermultiplet moduli space in the absence of NS5-brane instantons considered in the next section.

3. D-instantons and hypermultiplet moduli space

3.1. HM moduli space in type IIA string theory

Our aim is to investigate the HM moduli space of type-IIA string theory compactified on a CY threefold. It comprises $h_{2,1}(X) + 1$ hypermultiplets, which include the complex structure moduli $X^\Lambda = \int_{\gamma^\Lambda} \Omega$, $F_\Lambda = \int_{\gamma_\Lambda} \Omega$, the RR scalars $\zeta^\Lambda, \tilde{\zeta}_\Lambda$ representing the RR three-form integrated along a symplectic basis $(\gamma^\Lambda, \gamma_\Lambda)$ of A and B cycles in $H_3(X, \mathbb{Z})$, the four-dimensional dilaton $e^\phi = 1/g_{(4)}^2$ and the Neveu–Schwarz (NS) axion σ, dual to the Neveu–Schwarz two-form B in four dimensions. Whereas X^Λ provide a set of homogeneous coordinates for complex structure deformations, they may be traded for the inhomogeneous coordinates $z^a = X^a/X^0$.

To describe the HM moduli space, we will use the twistor approach from the previous section. This means that we should provide a covering of $\mathbb{C}P^1$, an associated set of transition functions and a set of anomalous dimensions. These data allow in principle one to compute the contact twistor lines (2.26), the contact potential (2.30) and to derive the metric. The coordinates appearing as free parameters of these solutions can be related to the physical fields of type-IIA string theory using considerations of symplectic invariance [23].

For the perturbative metric, which was initially obtained via the projective superspace and superconformal quotient [13–15, 28], such a formulation was given in [11]. At tree level, the HM moduli space is determined via the 'c-map' construction [29, 30] from the moduli space of complex structure deformations and therefore it is completely characterized by the

holomorphic prepotential $F(X)$. It specifies the transition functions $H^{[\pm 0]}$ corresponding to the covering of the twistor space $\mathcal{Z}$ by three patches: two patches $\mathcal{U}_+, \mathcal{U}_-$ are open discs centered around $\mathbf{z} = 0$ and $\mathbf{z} = \infty$ and a third patch $\mathcal{U}_0$ covers the rest of $\mathbb{C}P^1$ (see the next subsection for precise formulae). The one-loop correction is determined by the Euler number of CY, $\chi_X = 2(h^{1,1}(X) - h^{2,1}(X))$, and incorporated through a non-vanishing anomalous dimension c_α. It has been argued that there are no higher loop corrections [14].

The inclusion of non-perturbative effects into this framework was considered in [23]. It was suggested that every D2-instanton of charge $\gamma = (q_\Lambda, p^\Lambda)$, i.e. wrapping a three-cycle in the homology class $q_\Lambda \gamma^\Lambda - p^\Lambda \gamma_\Lambda \in H_3(X, \mathbb{Z})$, defines two 'BPS rays' on $\mathbb{C}P^1$ going from $\mathbf{z} = 0$ to $\mathbf{z} = \infty$. These rays introduce discontinuities in the twistor lines, thus requiring the introduction of additional patches. The transition functions through the rays are determined by dilogarithm functions, $\mathrm{Li}_2(x) = \sum_{m=1}^\infty m^{-2} x^m$, of symplectic invariant combinations of twistor lines, whereas the weights of instanton contributions were argued to be given by generalized Donaldson–Thomas invariants found in [24].[1]

However, these instanton corrections were analyzed only in the linear approximation. Although a proposal for the transition functions describing the exact twistor space has also been given, we will see that it was not quite precise and complete. In particular, only transition functions through the BPS rays were proposed, but even they were oversimplified. Below we present such an exact complete description providing a full consistent set of transition functions, which are argued to be valid at all orders in the instanton expansion. Our argumentation is based on symplectic invariance and it will be discussed in detail in the end of this section when all essential properties of the construction are already deduced. The construction itself represents the core of this section. First, we show how it is realized in the case of a single D2-instanton, where all relevant quantities can be computed explicitly, and then we generalize it to incorporate all instantons. Moreover we find that for a subset of all D-instantons consisting of 'mutually local states' there is an explicit representation of the contact twistor lines similar to the single-charge case.

3.2. The case of a single charge

3.2.1. Twistor space in the presence of one D2-instanton. In this subsection we consider the situation when the HM moduli space is affected by only one D2-instanton of charge $\gamma = (q_\Lambda, p^\Lambda)$. We give an improved version of the twistor space suggested in [23] and compute exact contact twistor lines and the contact potential.

First we introduce the following covering of $\mathbb{C}P^1$ (see figure 1, left). As we mentioned above, each charge vector γ defines a pair of 'BPS rays' $\ell_{\pm\gamma}$ on $\mathbb{C}P^1$ going between the north and south poles. They are restricted to lie in the hemispheres $V_{\pm\gamma}$ defined by

$$V_\gamma = \{\mathbf{z} : \mathrm{Im}(Z(\gamma)/\mathbf{z}) < 0\}, \tag{3.1}$$

where $Z(\gamma)$ is the normalized central charge function on $H_3(X, \mathbb{Z})$,

$$Z(\gamma) \equiv \frac{q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)}{\sqrt{K(z, \bar{z})}} \tag{3.2}$$

---

[1] A similar picture was suggested also in [25] in the context of $D = 3, \mathcal{N} = 4$ supersymmetric gauge theories obtained by compactifying $D = 4, \mathcal{N} = 2$ theories on a circle. In this case, the moduli space is HK and corrected by instantons representing 4D BPS solitons winding around the compactification circle. In fact, the two moduli spaces are very close not only at the qualitative level, but also quantitatively since the D-instantons are described in both cases by essentially the same transition functions.
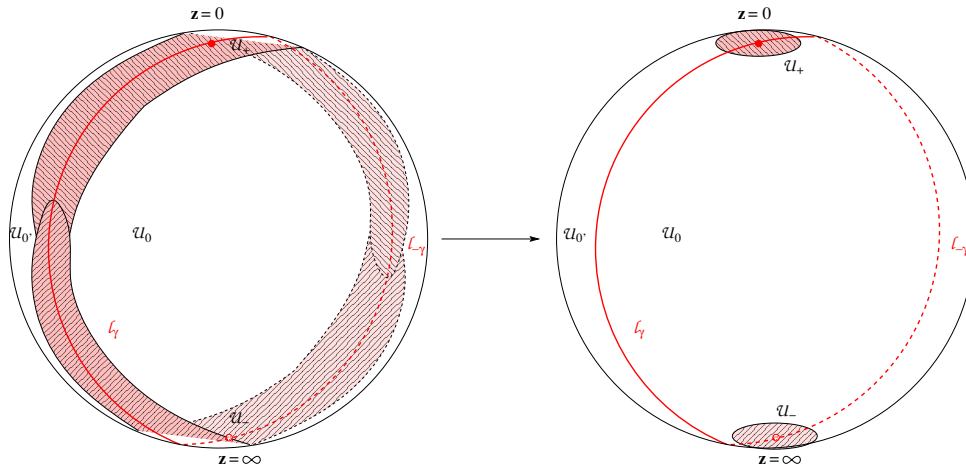
Figure 1. Two coverings of $\mathbb{C}P^1$. The covering on the left is at the basis of our construction of the HM twistor space affected by one D-instanton. The covering on the right is obtained in the limit where the strips $\mathcal{U}_\pm$ go to zero width along the meridians $\ell_{\pm\gamma}$, while maintaining a non-zero size at the north and south poles.

with $K(z, \bar{z}) = -2 \operatorname{Im}(\bar{z}^\Lambda F_\Lambda(z))$. For convenience, we can fix the position of the rays at the middle of the hemispheres

$$\ell_\gamma = \{\mathbf{z} : Z(\gamma)/\mathbf{z} \in \mathrm{i}\mathbb{R}^-\}. \tag{3.3}$$

Then we cover $\mathbb{C}P^1$ by four patches: the first patch $\mathcal{U}_+$ surrounds the north pole and extends along the rays $\ell_{\pm\gamma}$ down to the equator. The second patch $\mathcal{U}_-$ surrounds the south pole and similarly extends halfway along $\ell_{\pm\gamma}$, with a non-vanishing intersection with $\mathcal{U}_+$. The rest of $\mathbb{C}P^1$ consists of two connected parts covered by two patches $\mathcal{U}_0$ and $\mathcal{U}_{0'}$, which overlap with $\mathcal{U}_+$ and $\mathcal{U}_-$ but stay away from the contours $\ell_{\pm\gamma}$.

Thus, one has to specify transition functions corresponding to the overlaps of $\mathcal{U}_0$ and $\mathcal{U}_{0'}$ with $\mathcal{U}_\pm$. The transition function between $\mathcal{U}_+$ and $\mathcal{U}_-$ can then be determined by composing the previous ones. We suggest that the twistor space incorporating the effect of one D-instanton is produced by the following set

$$
\begin{aligned}
H^{[+0]} &= H^{[+0']} = \frac{\mathrm{i}}{2}\left[F(\xi_{[+]}) + \mathcal{G}_\gamma - \frac{1}{2}q_\Lambda p^\Lambda (\mathcal{G}'_\gamma)^2\right], \\
H^{[-0]} &= H^{[-0']} = \frac{\mathrm{i}}{2}\left[\bar{F}(\xi_{[-]}) - \mathcal{G}_\gamma - \frac{1}{2}q_\Lambda p^\Lambda (\mathcal{G}'_\gamma)^2\right]
\end{aligned}
\tag{3.4}
$$

and the only non-vanishing anomalous dimension is $c_\alpha = \chi_X/(96\pi)$. Here we introduced the function

$$\mathcal{G}_\gamma(\Xi_\gamma) = \frac{\mathrm{i}n_\gamma}{4\pi^3} \int_0^{-\mathrm{i}\infty} \frac{\Xi \, \mathrm{d}\Xi}{\Xi_\gamma^2 - \Xi^2} \operatorname{Li}_2(\mathrm{e}^{-2\pi \mathrm{i}\Xi}), \tag{3.5}$$

$\mathcal{G}'_\gamma$ denotes its derivative, and the argument Ξ_γ is related to the arguments of the transition functions through the transcendental equation

$$\Xi_\gamma^{[\pm 0]} \equiv q_\Lambda \xi_{[\pm]}^\Lambda + 2\mathrm{i}p^\Lambda \tilde{\xi}_\Lambda^{[0]} = \Xi_\gamma \mp q_\Lambda p^\Lambda \mathcal{G}'_\gamma(\Xi_\gamma). \tag{3.6}$$

The coefficients n_γ are supposed to coincide with the generalized Donaldson–Thomas invariants and for vanishing p^Λ can be related to the genus zero Gopakumar–Vafa invariants.

The holomorphic prepotential encodes the tree level part of the hypermultiplet metric, the anomalous dimension gives rise to the one-loop correction, and the function $\mathcal{G}_\gamma$ incorporates the effect of D2-instanton of charge γ. In the linear approximation, where one neglects the quadratic term in $\mathcal{G}_\gamma$, the transition functions (3.4) coincide with those proposed in [23]. Here the proposal is extended to all orders. In sections 3.2.2 and 3.4.1 we show that this extension is favored by symplectic invariance.

The transition functions (3.4) are designed in such a way in order to produce discontinuities in $\xi^\Lambda_{[0]}$, $\tilde{\xi}^{[0]}_\Lambda$ along the BPS rays given by simple $\log(1 - e^{-2\pi i \Xi_\gamma})$. Indeed, when one evaluates derivatives entering the gluing conditions (2.23), one should take into account that this amounts to differentiating the instanton contribution in (3.4) with respect to $\Xi^{[\pm 0]}_\gamma$, whereas the function $\mathcal{G}_\gamma$ depends on Ξ_γ. Therefore, one must use the relation (3.6), which ensures the following important property

$$\frac{\mathrm{d}}{\mathrm{d}\Xi^{[\pm 0]}_\gamma}\left(\mathcal{G}_\gamma \mp \frac{1}{2}q_\Lambda p^\Lambda(\mathcal{G}'_\gamma)^2\right) = \mathcal{G}'_\gamma(\Xi_\gamma). \tag{3.7}$$

The function $\mathcal{G}'_\gamma$ has the required discontinuity when Ξ_γ crosses the integration contour and one can adjust the contour in such way that in the **z**-plane, at least near the poles, it goes along $\ell_{\pm\gamma}$.

This consideration shows that the quadratic term is needed in order to take into account the difference in the arguments of the transition functions and of the function $\mathcal{G}_\gamma$. It could be avoided if one takes $\mathcal{G}_\gamma$ to be dependent directly of $\Xi^{[\pm 0]}_\gamma$. However, as will become clear later, this would spoil the symplectic invariance. Therefore, $\mathcal{G}_\gamma$ must depend on Ξ_γ defined through (3.6), which will turn out to be given by a symplectic invariant combination of twistor lines (see (3.15) below). As a result, the transition functions generate a twistor space carrying a representation of the symplectic group and allow an *exact* solution for the twistor lines (almost, i.e. up to quadratic terms appearing in the relation between the NS axion and the parameters of the twistor lines) coinciding with the linear approximation found in [23].

In order to present this solution, it will be convenient to work in the patch $\mathcal{U}_0$, where we expect the presence of symplectic invariance [11]. Besides, it is convenient to define the following combinations

$$\rho_\Lambda \equiv -2\mathrm{i}\tilde{\xi}^{[0]}_\Lambda, \qquad \tilde{\alpha} \equiv 4\mathrm{i}\alpha^{[0]} + 2\mathrm{i}\tilde{\xi}^{[0]}_\Lambda \xi^\Lambda_{[0]}. \tag{3.8}$$

Then we claim that the transition functions (3.4) lead to the following twistor lines

$$\xi^\Lambda_{[0]} = \zeta^\Lambda + \mathcal{R}(\mathbf{z}^{-1}z^\Lambda - \mathbf{z}\bar{z}^\Lambda) + \frac{n_\gamma}{8\pi^2}p^\Lambda \mathcal{I}^{(1)}_\gamma(\mathbf{z}), \tag{3.9a}$$

$$\rho_\Lambda = \tilde{\zeta}_\Lambda + \mathcal{R}(\mathbf{z}^{-1}F_\Lambda - \mathbf{z}\bar{F}_\Lambda) + \frac{n_\gamma}{8\pi^2}q_\Lambda \mathcal{I}^{(1)}_\gamma(\mathbf{z}), \tag{3.9b}$$

$$\tilde{\alpha} = \sigma + \mathcal{R}(\mathbf{z}^{-1}W - \mathbf{z}\bar{W}) + \frac{\mathrm{i}\chi_X}{24\pi}\log\mathbf{z} + \frac{\Xi_\gamma}{\pi^2}(\mathbf{z}^{-1}W_\gamma + \mathbf{z}\bar{W}_\gamma)\mathcal{K}_\gamma$$
$$+ \frac{\mathrm{i}n_\gamma}{8\pi^2}\left[\frac{1}{\pi\mathrm{i}}\mathcal{I}^{(2)}_\gamma(\mathbf{z}) + \left(\Theta_\gamma + \mathbf{z}^{-1}W_\gamma - \mathbf{z}\bar{W}_\gamma\right)\mathcal{I}^{(1)}_\gamma(\mathbf{z})\right], \tag{3.9c}$$

where we defined the type-IIA fields

$$\mathcal{R}z^\Lambda = Y^\Lambda, \qquad \zeta^\Lambda \equiv A^\Lambda, \qquad \tilde{\zeta}_\Lambda \equiv B_\Lambda + A^\Sigma \operatorname{Re}F_{\Lambda\Sigma} + \frac{n_\gamma}{2\pi^2}p^\Sigma \operatorname{Im}F_{\Lambda\Sigma}\mathcal{K}_\gamma,$$

$$\sigma \equiv -2B_\alpha - \zeta^\Lambda\tilde{\zeta}_\Lambda + \left(\zeta^\Lambda\zeta^\Sigma - \frac{n_\gamma^2}{4\pi^4}p^\Lambda p^\Sigma\mathcal{K}_\gamma^2\right)\operatorname{Re}F_{\Lambda\Sigma} \tag{3.10}$$
$$+ \frac{n_\gamma}{\pi^2}p^\Lambda\zeta^\Sigma\operatorname{Im}F_{\Lambda\Sigma}\mathcal{K}_\gamma + \frac{n_\gamma^2}{4\pi^4}q_\Lambda p^\Lambda\mathcal{K}_\gamma^2,$$

and introduced

$$W(z) \equiv F_\Lambda(z)\zeta^\Lambda - z^\Lambda \tilde{\zeta}_\Lambda. \tag{3.11}$$

$$W_\gamma \equiv \mathcal{R}(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)), \qquad \Theta_\gamma \equiv q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda, \tag{3.12}$$

as well as

$$\mathcal{K}_\gamma \equiv \sum_{m=1}^\infty \frac{1}{m} \sin(2\pi m \Theta_\gamma) K_0(4\pi m |W_\gamma|),$$

$$\mathcal{I}_\gamma^{(\nu)}(\mathbf{z}) \equiv \sum_{m=1}^\infty \sum_{s=\pm 1} \frac{s^\nu}{m^\nu} e^{-2\pi i s m \Theta_\gamma} \int_0^\infty \frac{dt}{t} \frac{t - \epsilon_\gamma s i \mathbf{z}}{t + \epsilon_\gamma s i \mathbf{z}} e^{-2\pi m |W_\gamma|(t^{-1}+t)}, \tag{3.13}$$

with $\epsilon_\gamma = e^{-i \arg W_\gamma}$. Here $\mathcal{R}$ is a real field which can be traded for the four-dimensional dilaton ϕ coinciding with the contact potential. Their relation is given below in (3.18). In the patch $\mathcal{U}_{0'}$ the twistor lines are given by the same formulae, although their analytic continuation to $\mathcal{U}_0$ does not coincide with (3.9) due to the discontinuities picked up by $\mathcal{I}_\gamma^{(\nu)}(\mathbf{z})$. In $\mathcal{U}_\pm$ the twistor lines can be obtained by applying the gluing conditions (2.23) with $H^{[ij]}$ from (3.4).

Since the twistor lines satisfy all gluing conditions by construction, to prove that they follow from (2.23), it is sufficient to show that they are regular everywhere except for the singularities allowed in (2.26). The regularity in the patches $\mathcal{U}_0$ and $\mathcal{U}_{0'}$ is evident because the only singularities of $\mathcal{I}_\gamma^{(\nu)}(\mathbf{z})$ are two cuts from $\mathbf{z} = 0$ to $\mathbf{z} = \infty$, which belong to $\mathcal{U}_+ \cup \mathcal{U}_-$. Thus, it remains to check the analytic structure in the patches $\mathcal{U}_\pm$.

For this purpose, note that, due to (3.7), the analytic structure (besides simple poles at $\mathbf{z} = 0, \infty$) of $\xi_{[\pm]}^\Lambda$ and $\tilde{\xi}_\Lambda^{[\pm]}$ is determined by the following combination[2]

$$\mathcal{I}_\gamma^{(1)}(\mathbf{z}) \pm 4 \left[\int_0^{-i\infty} \frac{\Xi_\gamma \, d\Xi}{\Xi_\gamma^2 - \Xi^2} \log(1 - e^{-2\pi i \Xi}) + \frac{\pi}{12 i \Xi_\gamma} \right]. \tag{3.14}$$

To analyze it, we need to know the dependence of Ξ_γ on $\mathbf{z}$. It can be found from (3.7) and (3.6), which imply

$$\Xi_\gamma = q_\Lambda \xi_{[0]}^\Lambda - p^\Lambda \rho_\Lambda. \tag{3.15}$$

Using the explicit solution (3.9), one therefore finds

$$\Xi_\gamma = \Theta_\gamma + \mathbf{z}^{-1} W_\gamma - \mathbf{z} \bar{W}_\gamma \tag{3.16}$$

and thus Ξ_γ does not contain instanton corrections. Now it is easy to check that although both $\mathcal{I}_\gamma^{(1)}(\mathbf{z})$ and the integral over Ξ have two cuts starting from $\mathbf{z} = 0$ ($\mathbf{z} = \infty$), their discontinuities cancel each other. As a result, (3.14) defines a meromorphic function in the neighborhood of the north (south) pole.

Finally, it remains to analyze $\alpha^{[\pm]}$. Using (3.8) and the gluing conditions (2.23), one obtains

$$4 i \alpha^{[\pm]} = \tilde{\alpha} - 2 i \xi_{[\pm]}^\Lambda \tilde{\xi}_\Lambda^{[\pm]} \pm (2\mathcal{G}_\gamma - \Xi_\gamma \mathcal{G}_\gamma'). \tag{3.17}$$

Again, it is easy to check that the discontinuities of the last term cancel the discontinuities in $\tilde{\alpha}$ generated by the terms in the last line of (3.9c). Besides, the simple poles at $\mathbf{z} = 0$ ($\mathbf{z} = \infty$), which are present in (3.9c), are all canceled by the second term in (3.17). Thus, the only singularity of $\alpha^{[\pm]}$ is given by the simple logarithmic term $\log \mathbf{z}$, in accordance with (2.26).

---

[2] $\tilde{\xi}_\Lambda^{[+]}$ contains an additional non-trivial term $F_\Lambda(\xi_{[+]})/(2i)$. But since we prove that $\xi_{[+]}^\Lambda$ is meromorphic in $\mathcal{U}_+$, this term is also meromorphic.

This completes the proof of our solution (3.9). In appendix B we also give a direct derivation of this solution from the integral representation (2.26).

The contact potential $\Phi_{A/B} = \phi$ corresponding to the solution (3.9) coincides with the result found in the linear instanton approximation and is given by

$$e^\phi = \frac{\mathcal{R}^2}{4} K(z, \bar{z}) + \frac{\chi_X}{192\pi} + \frac{n_\gamma}{4\pi^2} \sum_{m>0} \frac{|W_\gamma|}{m} \cos(2\pi m \Theta_\gamma) K_1(4\pi m |W_\gamma|). \tag{3.18}$$

Through (2.28) this result encodes the Kähler potential on $\mathcal{Z}$.

As we claimed above, the contact twistor lines (3.9) and the contact potential (3.18) are compatible with the action of the symplectic group. Indeed, if one simultaneously transforms the type-IIA fields and the charge vector $\gamma = (q_\Lambda, p^\Lambda)$, then $\xi_{[0]}^\Lambda$ and ρ_Λ form a symplectic vector, whereas $\tilde{\alpha}, \phi$ and the variable Ξ_γ are symplectic invariants.

3.2.2. Transition functions through BPS rays. It is instructive to find the transition function between the patches $\mathcal{U}_0$ and $\mathcal{U}_{0'}$. Although they do not intersect, we can define the corresponding transition function through the composition law of elementary contact transformations. In the particular case where the transition functions $H^{[ij]}$ are independent of $\alpha^{[j]}$, this law and the inverse transform are given by

$$\begin{aligned} H^{[ij]} &= H^{[ik]} + H^{[kj]} + \partial_{\xi_{[k]}^\Lambda} H^{[kj]} \partial_{\tilde{\xi}_\Lambda^{[k]}} H^{[ik]}, \\ H^{[ji]} &= -H^{[ij]} + \partial_{\xi_{[i]}^\Lambda} H^{[ij]} \partial_{\tilde{\xi}_\Lambda^{[j]}} H^{[ij]}, \end{aligned} \tag{3.19}$$

where, if necessary, the domain of definition of transition functions is extended by means of analytical continuation. Combining these equations, one finds

$$H^{[ij]} = -H^{[ki]} + H^{[kj]} + \partial_{\xi_{[k]}^\Lambda} (H^{[ki]} - H^{[kj]}) \partial_{\tilde{\xi}_\Lambda^{[k]}} H^{[ki]}. \tag{3.20}$$

The latter equation can already be applied to our twistor space. Let us specialize it for $i = 0, j = 0', k = +$. In fact, one can introduce two transition functions relating $\mathcal{U}_0$ and $\mathcal{U}_{0'}$ because there are two inequivalent ways to perform the necessary analytical continuation, either through ℓ_γ or through $\ell_{-\gamma}$. We attach the corresponding index to $H^{[00']}$ to distinguish these two cases. Then it is easy to show that

$$H^{[00']\pm} = \frac{\mathrm{i}}{2} \left(\pm G_{\pm\gamma} - \frac{1}{2} q_\Lambda p^\Lambda (G'_{\pm\gamma})^2 \right), \tag{3.21}$$

where

$$G_\gamma(\Xi_\gamma) = \frac{n_\gamma}{(2\pi)^2} \mathrm{Li}_2(e^{-2\pi\mathrm{i}\Xi_\gamma}) \tag{3.22}$$

and Ξ_γ is expressed through the arguments of the transition functions by means of

$$\Xi_\gamma^{[00']} \equiv q_\Lambda \xi_{[0]}^\Lambda + 2\mathrm{i} p^\Lambda \tilde{\xi}_\Lambda^{[0']} = \Xi_\gamma - q_\Lambda p^\Lambda G'_\gamma(\Xi_\gamma). \tag{3.23}$$

The transition functions (3.21) can be considered as describing a simplified version of our twistor space. This simplified version appears if one shrinks $\mathcal{U}_\pm$ along the contours $\ell_{\pm\gamma}$ and reduces them to small discs around the north and south poles of $\mathbb{C}P^1$, as shown on the right of figure 1. As a result, the two patches, $\mathcal{U}_0$ and $\mathcal{U}_{0'}$, have two disconnected common boundaries and the transition functions through them are given by $H^{[00']\pm}$. Such a description is a very convenient setup to perform calculations and it will be used in the next subsection to generalize the present construction in the case of several D-instantons.

In [23] a proposal for the exact contact transformations through the BPS rays has been advocated. In our notation it says that $H^{[00']\pm} = \pm \frac{\mathrm{i}}{2} G_{\pm\gamma} \left(\Xi_\gamma^{[00']} \right)$. The result (3.21) coincides with that proposal only in the linear approximation. At higher orders the difference is twofold.

First, the transition functions (3.21) contain an additional quadratic term proportional to the non-invariant combination of charges $q_\Lambda p^\Lambda$. Second, the argument of G_γ does not coincide with the combination of twistor lines $\Xi_\gamma^{[00']}$. As we explained below (3.7), these two differences are related and originate from the requirement that the discontinuities along the BPS rays be simple logarithmic functions of the symplectic invariant Ξ_γ. In turn, this is needed for symplectic invariance and to ensure that the linear approximation is exact. It must be exact for any single charge γ because, as mentioned in section 1, an appropriate symplectic transformation maps it to the form $(q'_\Lambda, 0)$ corresponding to a D-brane wrapping an A-cycle, which does possess this property. On the other hand, for the linear approximation to be exact, it is necessary that the argument of G_γ does not receive instanton corrections. This is true for Ξ_γ (3.16), but it is not true for $\Xi_\gamma^{[00']}$. Thus, we may claim that the proposal of [23] is not consistent with symplectic invariance, whereas our proposal is consistent, as follows from the explicit results for the contact twistor lines. An additional argument supporting this statement and formulated directly at the level of transition functions will be given in section 3.4.1.

Note that although G_γ coincides with the discontinuity of $\mathcal{G}_\gamma$, this is not true for the whole transition function: $H^{[00']\pm}$ cannot be obtained as a discontinuity of $H^{[+0]}$ due to the quadratic term. The correct way to obtain (3.21) is by means of the composition rule (3.20) as described above.

3.3. Inclusion of all instantons

3.3.1. Transition functions through BPS rays and twistor lines.
Let us include several instantons into consideration. The starting point will be the transition function (3.21) through a BPS ray representing the contribution of one instanton. Every instanton gives rise to two such rays on $\mathbb{C}P^1$, defined in (3.3) by the phase of its central charge. Following [23, 25], we assume that across every BPS ray the twistor lines experience contact transformations generated by functions (3.21).

More precisely, let $\{\gamma_a\}_{a=1}^N$ be a set of charges under consideration. We assume that the phases of their central charges $Z(\gamma_a)$ are all different. Then the BPS rays $\ell_{\pm\gamma_a}$ split $\mathbb{C}P^1$ into $2N$ sectors (see figure 2). For convenience we introduce additional N charges, which are the opposite of the initial ones, and we order all $2N$ charges, labeled by $i = 1, \ldots, 2N$, in accordance with decreasing of the phase of $Z(\gamma_i)$. Furthermore, the sector bounded by $\ell_{\gamma_{i-1}}$ and ℓ_{γ_i} will be denoted by $\mathcal{U}_i$ and we define

$$\Xi_{\gamma_k}^{[ij]} \equiv q_{k,\Lambda}\xi_{[i]}^\Lambda + 2\mathrm{i}p_k^\Lambda\tilde{\xi}_\Lambda^{[j]}. \tag{3.24}$$

With these definitions the transition functions through the BPS rays are

$$H^{[ii+1]} = \frac{\mathrm{i}}{2}\left(G_{\gamma_i} - \frac{1}{2}q_{i,\Lambda}p_i^\Lambda(G'_{\gamma_i})^2\right), \tag{3.25}$$

where $G_{\gamma_i}(\Xi_{\gamma_i})$ with $\Xi_{\gamma_i} \equiv \Xi_{\gamma_i}^{[ii]}$ is defined in (3.22). They generate the contact transformations expressed by the following relations between the contact twistor lines

$$\xi_{[j]}^\Lambda = \xi_{[i]}^\Lambda + \sum_{k=i}^{j-1} p_k^\Lambda G'_{\gamma_k}, \qquad -2\mathrm{i}\tilde{\xi}_\Lambda^{[j]} = -2\mathrm{i}\tilde{\xi}_\Lambda^{[i]} + \sum_{k=i}^{j-1} q_{k,\Lambda} G'_{\gamma_k}. \tag{3.26}$$

Although the gluing conditions (3.26) are linear in the functions G_{γ_i}, the resulting twistor space is much more complicated than the twistor space with a single instanton presented in section 3.2. In particular, the linear instanton approximation is no longer exact. The easiest way to see this is to consider Ξ_{γ_i} in this approximation. Substituting the results (3.9), where
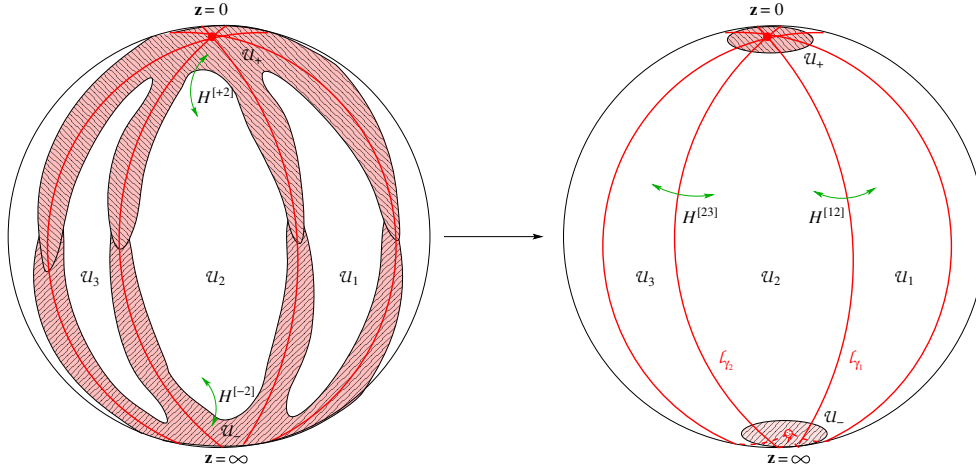
Figure 2. Coverings of $\mathbb{C}P^1$ in the presence of several instantons. The left picture defines a covering consisting of two patches around the poles and extending along the BPS rays and $2N$ patches filling the remaining holes, where N is the number of different charges. The right picture is a limit of the left one and is more suitable for the analysis of the twistor space.

the instanton contributions should contain the sum over all charges, into the definition (3.24), one obtains (3.16) plus the following contribution

$$\frac{1}{8\pi^2} \sum_j n_{\gamma_j} \langle \gamma_i, \gamma_j \rangle \mathcal{I}_{\gamma_j}^{(1)}(\mathbf{z}), \tag{3.27}$$

where

$$\langle \gamma_1, \gamma_2 \rangle = q_{1,\Lambda} p_2^\Lambda - q_{2,\Lambda} p_1^\Lambda \tag{3.28}$$

defines a symplectic invariant scalar product on the lattice of charges. Thus, the instantons affect the variables, which the transition functions depend on. As a result, their contributions propagate further and destroy the one-charge exact solution.

An exact solution can nevertheless be constructed by a perturbative approach described in appendix A.2. For $\xi^\Lambda, \tilde{\xi}_\Lambda$, it was analyzed in [25] where similar integral equations for the twistor lines of a HK space appeared. It is easy to see that it can be written as

$$\xi_{[i]}^\Lambda = \zeta^\Lambda + \mathcal{R}(\mathbf{z}^{-1} z^\Lambda - \mathbf{z}\bar{z}^\Lambda) + \frac{1}{8\pi^2} \sum_j n_{\gamma_j} p_j^\Lambda \mathcal{J}_{\gamma_j}(\mathbf{z}),$$

$$-2\mathrm{i}\tilde{\xi}_\Lambda^{[i]} = \tilde{\xi}_\Lambda + \mathcal{R}(\mathbf{z}^{-1} F_\Lambda - \mathbf{z}\bar{F}_\Lambda) + \frac{1}{8\pi^2} \sum_j n_{\gamma_j} q_{j,\Lambda} \mathcal{J}_{\gamma_j}(\mathbf{z}), \tag{3.29}$$

where $\mathbf{z} \in \mathcal{U}_i$,

$$\mathcal{J}_{\gamma_i}(\mathbf{z}) = \int_{\ell_{\gamma_i}} \frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'} \frac{\mathbf{z} + \mathbf{z}'}{\mathbf{z} - \mathbf{z}'} \log(1 - \mathrm{e}^{-2\pi\mathrm{i}\Xi_{\gamma_i}(\mathbf{z}')}), \tag{3.30}$$

and $\Xi_{\gamma_i}(\mathbf{z})$ should be found as a solution of the following system of equations

$$\Xi_{\gamma_i}(\mathbf{z}) = \Theta_{\gamma_i} + \mathbf{z}^{-1} W_{\gamma_i} - \mathbf{z}\bar{W}_{\gamma_i} + \frac{1}{8\pi^2} \sum_{j \neq i} n_{\gamma_j} \langle \gamma_i, \gamma_j \rangle \int_{\ell_{\gamma_j}} \frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'} \frac{\mathbf{z} + \mathbf{z}'}{\mathbf{z} - \mathbf{z}'} \log\left(1 - \mathrm{e}^{-2\pi\mathrm{i}\Xi_{\gamma_j}(\mathbf{z}')}\right).$$

$$\tag{3.31}$$

The latter equations encode all non-trivialities of the problem. They can be analyzed perturbatively and their solution, represented by a set of variables $\Xi_{\gamma_i}(\mathbf{z})$, contains all orders of the instanton expansion.

One can also obtain a similar representation for the twistor line α which is a bit more complicated. By appropriately adjusting the considerations in appendix B, one finds the following result, written again for the combination $\tilde{\alpha}$ defined in (3.8),

$$\tilde{\alpha}^{[i]} = \sigma + \mathcal{R}(\mathbf{z}^{-1}W - \mathbf{z}\bar{W}) + \frac{\mathrm{i}\chi_X}{24\pi}\log\mathbf{z} - \frac{1}{4\pi^2}\sum_j n_{\gamma_j}\big(\mathbf{z}^{-1}W_{\gamma_j} + \mathbf{z}\bar{W}_{\gamma_j}\big)\mathcal{J}_{\gamma_j}(0)$$

$$+ \frac{1}{8\pi^2}\sum_j n_{\gamma_j}\left[\frac{\mathrm{i}}{\pi}\int_{\ell_{\gamma_j}}\frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'}\frac{\mathbf{z}+\mathbf{z}'}{\mathbf{z}-\mathbf{z}'}\mathrm{Li}_2(\mathrm{e}^{-2\pi\mathrm{i}\Xi_{\gamma_j}(\mathbf{z}')}) + \big(\Theta_{\gamma_j} + \mathbf{z}^{-1}W_{\gamma_j} - \mathbf{z}\bar{W}_{\gamma_j}\big)\mathcal{J}_{\gamma_j}(\mathbf{z})\right]$$

$$+ \frac{1}{64\pi^2}\sum_{j\neq k} n_{\gamma_j}n_{\gamma_k}\langle\gamma_j,\gamma_k\rangle\int_{\ell_{\gamma_j}}\frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'}\frac{\mathbf{z}+\mathbf{z}'}{\mathbf{z}-\mathbf{z}'}\log\big(1 - \mathrm{e}^{-2\pi\mathrm{i}\Xi_{\gamma_j}(\mathbf{z}')}\big)\mathcal{J}_{\gamma_k}(\mathbf{z}'). \tag{3.32}$$

3.3.2. Transition functions to the poles. To complete the definition of the twistor space, we need to provide two additional transition functions to the patches $\mathcal{U}_{\pm}$ around the poles of $\mathbb{C}P^1$. We can find them requiring that they lead to regular contact twistor lines in these patches, i.e. that they cancel singularities of (3.29) and (3.32). The main complication comes from the fact that the variables $\Xi_{\gamma_i}(\mathbf{z})$ defined by (3.31) acquire branch cuts along the BPS rays ℓ_{γ_j}, $j \neq i$.[3] As a result, the simple sum of one-charge contributions (3.4) no longer works because canceling the singularities of the initial twistor lines by means of integrals introduces other singularities through the dependence of $\Xi_{\gamma_i}(\mathbf{z})$. Therefore, a more elaborated construction is required.

The correct transition functions to the patch $\mathcal{U}_+$ may be given in terms of $\Xi_{\gamma_i}^+ \equiv \Xi_{\gamma_i}^{[++]}$ and the functions $\Sigma_{\gamma_i}(\Xi)$ defined by the following system of integral equations

$$\Sigma_{\gamma_i}(\Xi) = \frac{1}{8\pi^2}\sum_j n_{\gamma_j}\langle\gamma_i,\gamma_j\rangle\left[\int_0^{-\mathrm{i}\infty}\frac{2\Xi\,\mathrm{d}\Xi'}{\Xi_{\gamma_j}^+\Xi - \Xi_{\gamma_i}^+\Xi'}\log\big(1 - \mathrm{e}^{-2\pi\mathrm{i}(\Xi'-\Sigma_{\gamma_j}(\Xi'))}\big) + \mathcal{E}_j\right], \tag{3.33}$$

where the functions $\mathcal{E}_j$ satisfy

$$\mathcal{E}_j = \frac{\pi\mathrm{i}}{6\Xi_{\gamma_j}^+} + \frac{1}{2\pi^2}\sum_k n_{\gamma_k}\langle\gamma_j,\gamma_k\rangle\int_0^{-\mathrm{i}\infty}\mathcal{D}_{\gamma_j}\Xi\int_0^{-\mathrm{i}\infty}\frac{\Xi'\mathcal{D}_{\gamma_k}\Xi'}{\big(\Xi_{\gamma_k}^+\Xi - \Xi_{\gamma_j}^+\Xi'\big)^2}$$

$$+ \frac{1}{16\pi^2}\sum_{k\neq l}\frac{n_{\gamma_k}n_{\gamma_l}}{n_{\gamma_j}}\langle\gamma_k,\gamma_l\rangle\,\mathcal{E}_k\frac{\partial\mathcal{E}_l}{\partial\Xi_{\gamma_j}^+}, \tag{3.34}$$

and we introduced for convenience the measure

$$\mathcal{D}_\gamma\Xi = \log(1 - \mathrm{e}^{-2\pi\mathrm{i}(\Xi-\Sigma_\gamma(\Xi))})\,\mathrm{d}\Xi. \tag{3.35}$$

---

[3] These variables possess interesting monodromies around the north and south poles. Since every charge γ_a appears in the sum together with its opposite $-\gamma_a$, the total contribution of this pair of charges to the monodromy is linear in Ξ_{γ_a}. As a result, going around the poles in the clockwise direction, one obtains the following transformation

$$\Xi_{\gamma_i} \longrightarrow \Xi_{\gamma_i} - \sum_a n_{\gamma_a}\langle\gamma_i,\gamma_a\rangle\,\Xi_{\gamma_a} \quad \mathrm{mod}\, n \in \mathbb{Z}.$$

Taking into account that n_{γ_a} are expected to be integers, we see that the monodromy gives rise to a linear combination of Ξ_{γ_a} with integer coefficients and can be represented as a linear map on the charge lattice:

$$\begin{pmatrix} q_\Lambda \\ p^\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} q_\Lambda \\ p^\Lambda \end{pmatrix} - \sum_{\gamma'\in\Gamma/\mathbb{Z}_2} n_{\gamma'}\langle\gamma,\gamma'\rangle\begin{pmatrix} q'_\Lambda \\ p'^\Lambda \end{pmatrix},$$

where $\Gamma/\mathbb{Z}_2$ is the charge lattice modulo identification $\gamma \leftrightarrow -\gamma$.

Similar to the equations for Ξ_{γ_i} (3.31), they can be solved perturbatively by expanding in powers of instantons, which is also equivalent to the expansion in invariants n_{γ_i}.[4]

Since $\Xi_{\gamma_i}^+$ is by definition meromorphic around the north pole, the singularities in the above expressions may appear only from the integrals. However, the denominator in (3.33) and (3.34) can be approximated by $\mathbf{z}^{-1}\big(W_{\gamma_j}\Xi - W_{\gamma_i}\Xi'\big)$ which does not vanish if the phases of W_{γ_i} and W_{γ_j} are different. Thus, both $\mathcal{E}_j$ and $\Sigma_{\gamma_i}(\Xi)$ are regular functions in the patch $\mathcal{U}_+$. However, the latter function possesses an important property that for $\Xi = \Xi_{\gamma_i}^+$ it gives rise to

$$\Sigma_{\gamma_i}\big(\Xi_{\gamma_i}^+\big) = \frac{1}{8\pi^2} \sum_j n_{\gamma_j} \langle \gamma_i, \gamma_j \rangle \mathcal{S}_j, \tag{3.36}$$

where

$$\mathcal{S}_j = 2 \int_0^{-i\infty} \frac{\mathrm{d}\Xi}{\Xi_{\gamma_j}^+ - \Xi} \log\big(1 - \mathrm{e}^{-2\pi i(\Xi - \Sigma_{\gamma_j}(\Xi))}\big) + \mathcal{E}_j \tag{3.37}$$

has a branch cut starting from $\mathbf{z} = 0$. If we impose the relation

$$\Xi_{\gamma_i} = \Xi_{\gamma_i}^+ - \Sigma_{\gamma_i}\big(\Xi_{\gamma_i}^+\big), \tag{3.38}$$

then the discontinuity along this cut is the same as the discontinuity of the function $\mathcal{J}_{\gamma_i}$ (3.30). This is consistent with the regularity of $\Xi_{\gamma_i}^+$ in $\mathcal{U}_+$ because the relation (3.38) ensures that the corresponding singularities of $\Sigma_{\gamma_i}\big(\Xi_{\gamma_i}^+\big)$ cancel those of Ξ_{γ_i}. The functions $\mathcal{E}_j$ are not really important since they are regular and can be canceled by appropriate (although complicated) gauge transformation.

Now we are ready to define the transition functions. They are given by

$$H^{[+i]} = \frac{i}{2} F(\xi_{[+]}) - \frac{1}{16\pi^3} \sum_j n_{\gamma_j} \int_0^{-i\infty} \frac{\mathrm{d}\Xi}{\Xi_{\gamma_j}^+ - \Xi} \mathrm{Li}_2\big(\mathrm{e}^{-2\pi i(\Xi - \Sigma_{\gamma_j}(\Xi))}\big)$$
$$- \frac{i}{(16\pi^2)^2} \sum_{j\neq k} n_{\gamma_j} n_{\gamma_k} q_{j,\Lambda} p_k^{\Lambda} \mathcal{S}_j \mathcal{S}_k - \frac{i}{16\pi^2} \sum_j n_{\gamma_j} \int_0^{-i\infty} \frac{\mathcal{D}_{\gamma_j}\Xi}{\Xi_{\gamma_j}^+ - \Xi} \Sigma_{\gamma_j}(\Xi). \tag{3.39}$$

In appendix C we verify that they indeed define regular twistor lines in the patch $\mathcal{U}_+$ and they are consistent with the transition functions through the BPS rays (3.25). In a similar way one can construct the transition functions to the patch $\mathcal{U}_-$. For this it is sufficient to replace $F(\xi_{[+]})$ by $\bar{F}(\xi_{[-]})$, $\Xi_{\gamma_i}^+$ by $\Xi_{\gamma_i}^-$, and to flip the sign of n_{γ_j} in all above equations.

Once we know the full set of transition functions, it is possible to compute the contact potential coinciding with the four-dimensional dilaton. This calculation uses the property

$$\xi_{[\pm]}(\mathbf{z}) = \xi_{[i]}(\mathbf{z}) + O(\mathbf{z}^{\pm 2}), \tag{3.40}$$

which can be established from (C.5), (3.37) and (3.34). Then the general formula for (the constant part of) the contact potential (2.31) gives the following result

$$e^{\phi} = \frac{\mathcal{R}^2}{4} K(z, \bar{z}) + \frac{\chi_X}{192\pi} - \frac{i}{16\pi^2} \sum_j n_{\gamma_j} \int_{\ell_{\gamma_j}} \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}} \big(\mathbf{z}^{-1} W_{\gamma_j} - \mathbf{z}\bar{W}_{\gamma_j}\big) \log\big(1 - \mathrm{e}^{-2\pi i \Xi_{\gamma_j}(\mathbf{z})}\big). \tag{3.41}$$

---

[4] Note that n_{γ_j} in the denominator of the last term in (3.34) is canceled after substituting $\partial_{\Xi_{\gamma_j}^+} \mathcal{E}_l$.

3.3.3. Restriction to mutually local states.　　Let us note that if we restrict our attention to a sector in the charge lattice where all charges satisfy the following condition

$$\langle \gamma_i, \gamma_j \rangle = 0, \tag{3.42}$$

all above-mentioned complications, arising in the presence of several charges, disappear. Indeed, substituting it into (3.31), one finds that $\Xi_{\gamma_i}(\mathbf{z})$ do not contain instanton corrections as in the single-charge case. Moreover, this is true for all $\Xi_{\gamma_i}^{[jj]}$ which turn out to be independent of the label j,

$$\Xi_{\gamma_i}^{[jj]} = \Theta_{\gamma_i} + \mathbf{z}^{-1} W_{\gamma_i} - \mathbf{z} \bar{W}_{\gamma_i}. \tag{3.43}$$

As a result, the linear instanton approximation again becomes exact. The contact twistor lines are given by (3.29), where the function $\mathcal{J}_\gamma(\mathbf{z})$ may be replaced by the function $\frac{1}{2}\mathcal{I}_\gamma^{(1)}(\mathbf{z})$ (3.13),[5] and by (3.32) where the last term vanishes and the same replacement may be done. Similarly, the transition functions (3.39) considerably simplify and have the same structure (3.4) as for single charge: the last term is absent and the third term coincides with the product of derivatives of the second term.

　　Thus, the condition (3.42) gives rise to an exactly solvable sector. The exact twistor lines can be used to extract the metric on the HM moduli space affected by D2-instantons with charges satisfying this restriction, which we call by 'mutually local states'. Although this procedure is straightforward, we do not present here the final result since it has a quite involved form.

　　It is clear that a particular set of mutually local states is given by D2 branes wrapping only A-cycles which all have $p^\Lambda = 0$. This is a maximal possible set of such states and any other maximal set can be obtained by a symplectic transformation[6]. Due to this, it is not surprising that any such sector is exactly solvable. This is just a consequence of such solvability for A-type D2-instantons. Nevertheless, this still provides a non-trivial cross-check on our results.

3.4. Discussion

3.4.1. Symplectic invariance.　　Although the found twistor lines form a nice representation of the symplectic group, one can ask whether the conditions of symplectic invariance can be imposed directly on transition functions. Naively, one could expect that they should be invariant under symplectic transformations. But for the functions $H^{[\pm i]}$ this is not true even at the tree level due to the presence of the holomorphic prepotential. As for the transition functions through the BPS rays (3.25), they are not invariant due to the factor $q_\Lambda p^\Lambda$ in front of the additional quadratic term.

　　To understand what is going on, let us consider the gluing conditions (2.23). If in both patches $\mathcal{U}_i, \mathcal{U}_j$, the twistor lines $\xi^\Lambda, -2i\tilde{\xi}_\Lambda$ form a symplectic vector and $\tilde{\alpha}$ is invariant, the same property must hold for the appropriate combinations of derivatives of the transition function $H^{[ij]}$. However, it is difficult to convert this condition into a restriction on $H^{[ij]}$ itself. The reason is that the transition functions used so far relate coordinate systems in two different patches and therefore depend also on coordinates in these different patches. On the other hand, symplectic invariance holds only for quantities defined in a single patch.

　　It is well known that generically one cannot write a symplectomorphism or a contact transformation in terms of a function dependent only on initial coordinates. Let us nevertheless

---

[5]　More precisely, (3.43) ensures that $\mathcal{J}_\gamma(\mathbf{z}) - \mathcal{J}_{-\gamma}(\mathbf{z}) = \mathcal{I}_\gamma^{(1)}(\mathbf{z})$. Combining the contributions of charges γ and $-\gamma$, one obtains the simple factor $\frac{1}{2}$.
[6]　For the universal hypermultiplet [20, 31–35] this means that mutually local states can include only D-branes wrapping either the A or B cycle.

find the conditions under which this might be possible. If one writes the contact transformation (which induces a symplectomorphism in the $(\xi, \tilde{\xi})$-subspace) as

$$\xi^\Lambda_{[j]} = \xi^\Lambda_{[i]} - \partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]}, \qquad \tilde{\xi}^{[j]}_\Lambda = \tilde{\xi}^{[i]}_\Lambda + \partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]},$$

$$\alpha^{[j]} = \alpha^{[i]} + \hat{H}^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]} + \tfrac{1}{2} \partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]} \partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]},$$

(3.44)

where $\hat{H}^{[ij]}$ is a function of $\xi^\Lambda_{[i]}$ and $\tilde{\xi}^{[i]}_\Lambda$, and requires that the contact one-form $\mathcal{X}$ (2.20) be preserved, one finds the following condition on $\hat{H}^{[ij]}$

$$\partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]} d\big(\partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]}\big) = \partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]} d\big(\partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]}\big).$$

(3.45)

Assume for a moment that this condition is satisfied. Then the symplectic properties of the contact twistor lines ξ^Λ, $\tilde{\xi}_\Lambda$ require that $\big(\partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]}, 2i\partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]}\big)$ be a symplectic vector, whereas from the gluing condition on $\tilde{\alpha}$,

$$\tilde{\alpha}^{[j]} = \tilde{\alpha}^{[i]} + 2i\big(2\hat{H}^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} \hat{H}^{[ij]} - \tilde{\xi}^{[i]}_\Lambda \partial_{\tilde{\xi}^{[i]}_\Lambda} \hat{H}^{[ij]}\big),$$

(3.46)

one concludes that $\hat{H}^{[ij]}$ must be invariant under symplectic transformations. The latter condition automatically solves also the former. Thus, if it is possible to write the contact transformations between different patches in terms of transition functions dependent on coordinates in one patch, symplectic invariance simply requires the invariance of these functions.

Remarkably, condition (3.45) ensuring this possibility is satisfied by functions for which all dependence on the contact twistor lines is through the symplectic invariant combination Ξ_{γ_i}. Thus, there is a class of contact transformations for which the generating functions can be taken as in (3.44). In particular, we can define the following generating functions

$$\hat{H}^{[ii+1]}(\xi_{[i]}, \tilde{\xi}^{[i]}) = \frac{i}{2} G_{\gamma_i}(\Xi_{\gamma_i}),$$

(3.47)

which are explicitly symplectic invariant. It is easy to see that they generate the same contact transformations through the BPS rays as those generated by (3.25). This proves in another way that our construction respects symplectic invariance.

If one allows the transition functions to depend on several combinations $\Xi^{[ii]}_{\gamma_j}$ associated with different charges γ_j, the condition (3.45) is not satisfied unless the charges are mutually local (3.42). This is related to the phenomenon of wall crossing considered in the next subsection. Besides, the non-invariance of the tree-level part expressed through the holomorphic prepotential is not in contradiction with symplectic invariance because it appears in the transition functions to the patches $\mathcal{U}_\pm$ only. But the contact twistor lines in these patches do not form a representation of the symplectic group. Therefore, the above arguments cannot be applied to $\hat{H}^{[\pm i]}$.

The representation (3.47) is very nice since the non-invariant quadratic terms disappear and all considerations become particularly simple. Note however the presence of a quadratic term in the gluing condition for α (3.44), which was absent in (2.24). Thus, we traded quadratic terms in the transition functions for similar terms in the gluing conditions. It is possible to get rid of them everywhere if one simultaneously trades α for $\tilde{\alpha}$ since the gluing condition (3.46) for the latter is linear.

3.4.2. Wall crossing. In [25] a physical explanation for the so-called wall-crossing formula [24] has been suggested in the context of $\mathcal{N} = 2$ supersymmetric gauge theories. It has been interpreted as a condition on the moduli space metric to be continuous across the lines of marginal stability (LMS), where the BPS spectrum of single-particle states is known to jump (see e.g. [36–38]). The wall-crossing formula relates the single instanton contribution on one

side of the LMS to the multi-instanton contribution on the other side. This provides strong constraints on these contributions.

A similar phenomenon as LMS is known to take place in $\mathcal{N} = 2$ supergravity theories and in [23] it was suggested that the wall-crossing formula is relevant also in this case expressing the condition of regularity of the metric on the hypermultiplet moduli space. Here we would like to show that our construction is consistent with the wall crossing.

The wall-crossing condition requires that the 'generalized Donaldson–Thomas invariants' $\Omega(\gamma)$ defined in [24] must satisfy

$$\overset{\curvearrowleft}{\prod_{\substack{\gamma=n\gamma_1+m\gamma_2 \\ m>0, n>0}}} U_\gamma^{\Omega^-(\gamma)} = \overset{\curvearrowright}{\prod_{\substack{\gamma=n\gamma_1+m\gamma_2 \\ m>0, n>0}}} U_\gamma^{\Omega^+(\gamma)}, \tag{3.48}$$

where $\Omega^-(\gamma)$ and $\Omega^+(\gamma)$ denote the value of $\Omega(\gamma)$ on either side of the LMS where the phases of the central charges (3.2), $Z(\gamma_1)$ and $Z(\gamma_2)$, align. Here

$$U_\gamma \equiv \exp\left(\sum_{n=1}^\infty \frac{1}{n^2} e_{n\gamma}\right) \tag{3.49}$$

is a group element constructed from the generators e_γ of the following Lie algebra:

$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle e_{\gamma+\gamma'}. \tag{3.50}$$

Except for the sign $(-1)^{\langle \gamma, \gamma' \rangle}$, which can be absorbed into a redefinition of e_γ by a choice of 'quadratic refinement' [25], this is the algebra of infinitesimal symplectomorphisms on the complex torus $(\mathbb{C}^\times)^{2N}$.

In our case the LMS appear because the order of the BPS rays, determined by the phases of the central charges $Z(\gamma_i)$, is important. Despite apparent commutativity of the contact transformations (3.26), it is illusive because the exchange of two charges changes relations between the arguments Ξ_{γ_i} of the functions entering the transformations. To see this explicitly and to compare with the wall crossing, it is instructive to compute the composition of two transition functions (3.25) associated with charges γ_1 and γ_2. From the composition law (3.19), one finds

$$H^{[13]} = \frac{i}{2}\left[G_{\gamma_1} + G_{\gamma_2} - \frac{1}{2}\left(q_{1,\Lambda} G'_{\gamma_1} + q_{2,\Lambda} G'_{\gamma_2}\right)\left(p_1^\Lambda G'_{\gamma_1} + p_2^\Lambda G'_{\gamma_2}\right) + \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle G'_{\gamma_1} G'_{\gamma_2} \right]. \tag{3.51}$$

From this result and using the relations (3.26) between the coordinates in different patches, one can find the commutator of two transformations U_{γ_i} generated by (3.25) with charges γ_1 and γ_2. Expanding in powers of G_{γ_i}, at quadratic order one obtains

$$U_{\gamma_2}^{-1} U_{\gamma_1}^{-1} U_{\gamma_2} U_{\gamma_1} \approx -\langle \gamma_1, \gamma_2 \rangle G'_{\gamma_1} G'_{\gamma_2} + O\left(G_\gamma^3\right). \tag{3.52}$$

It is clear that if we identify $e^{-2\pi i \Xi_\gamma}$ with the elements e_γ, this commutation relation coincides with the commutator (3.50), which is on the basis of the wall-crossing formula. The construction from the previous subsection makes the comparison even more direct since then the transition functions (3.47) are identical to (the logarithm of) the group elements (3.49).

Finally, we note that the composition law (3.51) for two contact transformations through the BPS rays can easily be generalized to the case of n charges:

$$H^{[1n+1]} = \frac{i}{2}\left[\sum_i G_{\gamma_i} - \frac{1}{2}\left(\sum_i q_{i,\Lambda} G'_{\gamma_i}\right)\left(\sum_i p_i^\Lambda G'_{\gamma_i}\right) + \frac{1}{2}\sum_{i<j}\langle \gamma_i, \gamma_j \rangle G'_{\gamma_i} G'_{\gamma_j} \right]. \tag{3.53}$$

On the other hand, the above results imply that the contact transformations generated by the mutually local states of section 3.3.3 are commutative, which means that the LMS phenomenon does not occur in such a sector.

Acknowledgments

Appendix A. Details on the twistor description of HK spaces

A.1. Solution for the Lagrangian

Here we want to prove the formula (2.16) for the Lagrangian, i.e. that it ensures the conditions (2.15). For $\mathcal{L}$ given by this formula, by simple manipulations, one can establish the following relations

$$
\begin{aligned}
\mathcal{L}_{v^I} &= \partial_{v^I} u^J \oint_C \frac{d\zeta}{2\pi i \zeta^2} H_J - \partial_{v^I} \bar{u}^J \oint_C \frac{d\zeta}{2\pi i} H_J, \\
\mathcal{L}_{\bar{v}^I} &= \partial_{\bar{v}^I} u^J \oint_C \frac{d\zeta}{2\pi i \zeta^2} H_J - \partial_{\bar{v}^I} \bar{u}^J \oint_C \frac{d\zeta}{2\pi i} H_J, \\
\mathcal{L}_{x^I} &= \oint_C \frac{d\zeta}{2\pi i \zeta} H_J + \partial_{x^I} u^J \oint_C \frac{d\zeta}{2\pi i \zeta^2} H_J - \partial_{x^I} \bar{u}^J \oint_C \frac{d\zeta}{2\pi i} H_J, \\
\mathcal{L}_{\varrho_I} &= \frac{i}{2} \oint_C \frac{d\zeta}{2\pi i \zeta} H^J + \partial_{\varrho_I} u^J \oint_C \frac{d\zeta}{2\pi i \zeta^2} H^J - \partial_{\varrho_I} \bar{u}^J \oint_C \frac{d\zeta}{2\pi i} H^J,
\end{aligned}
\tag{A.1}
$$

where $\mathcal{L}$ on the lhs is considered as a function of $v^I, \bar{v}^I, x^I, \varrho_I$. Comparing this with the relations between derivatives

$$
\begin{aligned}
\partial_{v^I}|_{\bar{v},x,\varrho} &= \partial_{v^I} u^J \partial_{u^J} + \partial_{v^I} \bar{u}^J \partial_{\bar{u}^J}, \\
\partial_{\bar{v}^I}|_{v,x,\varrho} &= \partial_{\bar{v}^I} u^J \partial_{u^J} + \partial_{\bar{v}^I} \bar{u}^J \partial_{\bar{u}^J}, \\
\partial_{x^I}|_{v,\bar{v},\varrho} &= \partial_{x^I} + \partial_{x^I} u^J \partial_{u^J} + \partial_{x^I} \bar{u}^J \partial_{\bar{u}^J}, \\
\partial_{\varrho_I}|_{v,\bar{v},x} &= \partial_{\varrho_I} + \partial_{\varrho_I} u^J \partial_{u^J} + \partial_{\varrho_I} \bar{u}^J \partial_{\bar{u}^J},
\end{aligned}
\tag{A.2}
$$

one immediately concludes that the Lagrangian $\mathcal{L}$ satisfies the conditions (2.15). It is equivalent to the previous perturbative results [10] due to the relation

$$
\oint_C \frac{d\zeta}{2\pi i \zeta} (\hat{\eta}^I H_I - \hat{\mu}_I H^I) = (u^I - v^I) \partial_{u^I} \mathcal{L} + (\bar{u}^I - \bar{v}^I) \partial_{\bar{u}^I} \mathcal{L}.
\tag{A.3}
$$

The fact that the complex structures must satisfy the algebra of the quaternions, e.g.

$$
J^+ J^- = -\tfrac{1}{2}(\mathbf{1} + i J^3) \quad \leftrightarrow \quad (\omega^+)_{\alpha\gamma} K^{\gamma\bar{\delta}} (\omega^-)_{\bar{\delta}\bar{\beta}} = K_{\alpha\bar{\beta}},
\tag{A.4}
$$

imposes restrictions on the inverse metric. It is easy to find that it must be given by

$$
\begin{aligned}
K^{\bar{u}^I u^J} &= \mathcal{L}_{\varrho_I \varrho_J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L \varrho_J} - \mathcal{L}^{x^I x^J} - i \left[\mathcal{L}^{x^I x^K} \mathcal{L}_{x^K \varrho_J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^J} \right], \\
K^{\bar{u}^I w_J} &= -\mathcal{L}^{x^I x^K} \mathcal{L}_{x^K u^J} - i \left[\mathcal{L}_{\varrho_I u^J} - \mathcal{L}_{\varrho_I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L u^J} \right], \\
K^{\bar{w}_I u^J} &= -\mathcal{L}_{\bar{u}^I x^K} \mathcal{L}^{x^K x^J} + i \left[\mathcal{L}_{\bar{u}^I \varrho_J} - \mathcal{L}_{\bar{u}^I x^M} \mathcal{L}^{x^M x^N} \mathcal{L}_{x^N \varrho_J} \right], \\
K^{\bar{w}_I w_J} &= \mathcal{L}_{\bar{u}^I u^J} - \mathcal{L}_{\bar{u}^I x^K} \mathcal{L}^{x^K x^L} \mathcal{L}_{x^L u^J}.
\end{aligned}
\tag{A.5}
$$

This result requires that the Lagrangian satisfies some additional constraints in order to ensure that (A.5) is indeed the inverse of (2.19). These constraints are a consequence of the fact that $\mathcal{L}$ is given by a holomorphic function of only $2d + 1$ variables and generalize the well-known

constraints in the $\mathcal{O}(2)$ case [8], which were written recently also in the presence of linear perturbations [10].

A.2. Perturbative solution

The integral equations (2.9) and (2.26) are a very convenient starting point to find twistor lines by the perturbative method. This approach can be applied if the transition functions can be represented as

$$H^{[ij]} = \sum_{n=0}^{\infty} \lambda^n H^{[ij]}_{(n)}, \tag{A.6}$$

where $H^{[ij]}_{(0)}$ are all independent of $\mu^{[j]}_I$ and λ is an infinitesimal parameter. Then at the zeroth approximation, η^I are global $\mathcal{O}(2)$ multiplets which can be viewed as moment maps for the isometries along ϱ_I. At the next orders in the parameter λ however this structure is destroyed. Expanding in λ, the integral equations (2.9) lead to the following solution

$$\eta^I_{[i]} = \eta^I_{(0)} + \sum_{n=1}^{\infty} \lambda^n \eta^I_{(n)[i]}, \qquad \mu^{[i]}_I = \frac{\mathrm{i}}{2}\varrho_I + \sum_{n=0}^{\infty} \lambda^n \mu^{[i]I}_{(n)}, \tag{A.7}$$

where $\eta^I_{(0)}$ was defined in (2.10) and

$$\begin{aligned}
\eta^I_{(n)[i]} &= -\frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i}\zeta'} \frac{\zeta'^3 + \zeta^3}{\zeta\zeta'(\zeta' - \zeta)} \left[H^{[ij]I}_{(n)} + \cdots \right], \\
\mu^{[i]I}_{(n)} &= \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i}\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \left[H^{[ij]}_{(n)I} + \cdots + \eta^J_{(n)[i]} H^{[ij]}_{(0)IJ} \right],
\end{aligned} \tag{A.8}$$

where the transition functions are considered as functions of $\eta_{(0)}$, $\frac{\mathrm{i}}{2}\varrho + \mu^{[j]}_{(0)}$ and ζ', and the dots correspond to other terms of nth order appearing in the expansion of $H^{[ij]}$, which all depend only on $\eta^{[i]}_{(k)}, \mu^{[j]}_{(k)}, k < n$. Thus, this leads to a well-defined iteration procedure.

Appendix B. Derivation of contact twistor lines for single charge

In this appendix we demonstrate that the twistor lines (3.9) satisfy the integral equations (2.26).

First, we compute the functions entering the gluing conditions (2.23) and defined in (2.24). Due to (3.7), one has

$$\begin{aligned}
T^\Lambda_{[+0]} &= p^\Lambda \mathcal{G}'_\gamma(\Xi_\gamma), \qquad \tilde{T}^{[+0]}_\Lambda = \frac{\mathrm{i}}{2}(F_\Lambda(\xi_{[+]}) + q_\Lambda \mathcal{G}'_\gamma(\Xi_\gamma)), \\
\tilde{T}^{[+0]}_\alpha &= \frac{\mathrm{i}}{2}\left(-F(\xi_{[+]}) + \mathcal{G}_\gamma(\Xi_\gamma) - q_\Lambda \xi^\Lambda_{[0]} \mathcal{G}'_\gamma(\Xi_\gamma) + \frac{1}{2}q_\Lambda p^\Lambda (\mathcal{G}'_\gamma(\Xi_\gamma))^2 \right).
\end{aligned} \tag{B.1}$$

The corresponding functions associated with the pair of patches $\mathcal{U}_-$ and $\mathcal{U}_0$ can be obtained replacing $F(\xi_{[+]})$ by $\bar{F}(\xi_{[-]})$ and changing the sign of the function $\mathcal{G}_\gamma$. The explicit expression for $\mathcal{G}'_\gamma(\Xi_\gamma)$ is given by

$$\mathcal{G}'_\gamma(\Xi_\gamma) = -\frac{n_\gamma}{2\pi^2}\left[\int_0^{-\mathrm{i}\infty} \frac{\Xi_\gamma\,\mathrm{d}\Xi}{\Xi_\gamma^2 - \Xi^2} \log(1 - \mathrm{e}^{-2\pi\mathrm{i}\Xi}) + \frac{\pi}{12\mathrm{i}\Xi_\gamma} \right] \tag{B.2}$$

and Ξ_γ can be found in (3.16). Writing $\frac{2\Xi_\gamma}{\Xi_\gamma^2 - \Xi^2} = \frac{1}{\Xi_\gamma - \Xi} + \frac{1}{\Xi_\gamma + \Xi}$, it is clear that $\mathcal{G}'_\gamma$ considered as a function of $\mathbf{z}$, as well as the initial function $\mathcal{G}_\gamma$, is a sum of two functions each having
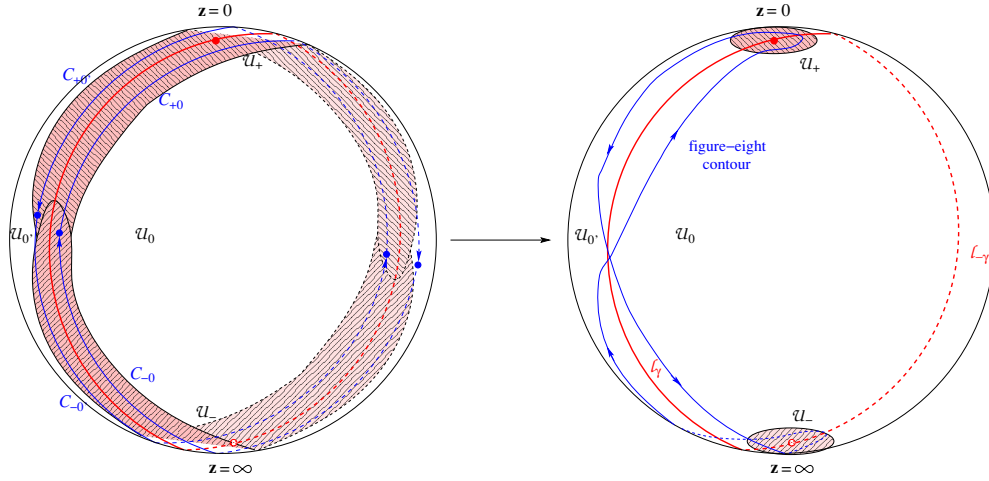
Figure B1. The contours on $\mathbb{C}P^1$. The left picture shows the covering of $\mathbb{C}P^1$ and the contours used in (B.3). On the right these contours are transformed into one figure-eight contour, which should be used to integrate one of the two terms in $\mathcal{G}'_\gamma$. The second term should be integrated along a similar figure-eight contour going around $\ell_{-\gamma}$.

two cuts going from $\mathbf{z} = 0$ and $\mathbf{z} = \infty$ to two zeros of Ξ_γ, which are supposed to lie outside $\mathcal{U}_+ \cup \mathcal{U}_-$.

Due to the presence of these cuts, the representation (2.26) for the twistor lines is not directly applicable because the contours around $\mathcal{U}_\pm$ cannot be closed. To overcome this problem, it is convenient to start with the following equivalent representation, exemplified here for $\tilde{\xi}^{[0]}_\Lambda$,[7]

$$\tilde{\xi}^{[0]}_\Lambda = \frac{\mathrm{i}}{2} B_\Lambda - \frac{1}{2} \sum_\pm \left[\oint_{C_{\pm 0}} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \tilde{T}^{[\pm 0]}_\Lambda(\mathbf{z}') + \oint_{C_{\pm 0'}} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \tilde{T}^{[\pm 0']}_\Lambda(\mathbf{z}') \right], \qquad (B.3)$$

where $\mathbf{z} \in \mathcal{U}_0$ and the contours are shown in figure B1 (left). Then we have to consider separately terms differing by the power of $\mathcal{G}_\gamma$. The terms independent of $\mathcal{G}_\gamma$, which are proportional to the holomorphic prepotential, are all meromorphic, having at most poles at $\mathbf{z} = 0$ and $\mathbf{z} = \infty$. Moreover, they coincide from two sides ($\mathcal{U}_0$ and $\mathcal{U}_{0'}$). Therefore, for these terms the contours can be deformed just to two small circles around the north and south poles of $\mathbb{C}P^1$, and the corresponding integrals are evaluated by residues. All terms linear in $\mathcal{G}_\gamma$, as noticed above, can be written as a linear combination of functions with two cuts and they have opposite signs in T's with indices $[\pm 0]$. Such a situation was analyzed in [10] where it was demonstrated that for each term the initial contours can be combined into one figure-eight contour going around $\mathbf{z} = 0$ and $\mathbf{z} = \infty$, as shown in figure B1 (right). The integrals around such contours reduce then to integrals along $\ell_{\pm \gamma}$ of the corresponding discontinuity. This analysis is sufficient to evaluate $\xi^\Lambda_{[0]}$ and $\tilde{\xi}^{[0]}_\Lambda$. It is easy to verify that the results indeed coincide with (3.9a) and (3.9b).

The calculation of $\alpha^{[0]}$ is more complicated due to the presence of terms quadratic in $\mathcal{G}_\gamma$, which should be treated with great care. Let us collect the contributions quadratic in

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[7] Note a sign difference with (2.26).

instantons, which come from the last two terms in $\tilde{T}_\alpha^{[\pm 0]}$. They are given by

$$\frac{\mathrm{i}}{4} q_\Lambda p^\Lambda \left((\mathcal{G}'_\gamma)^2 \mp \frac{n_\gamma}{4\pi^2} \mathcal{I}_\gamma^{(1)}(\mathbf{z}) \mathcal{G}'_\gamma \right) = \frac{\mathrm{i}}{4} q_\Lambda p^\Lambda \left[\left(\mathcal{G}'_\gamma \mp \frac{n_\gamma}{8\pi^2} \mathcal{I}_\gamma^{(1)}(\mathbf{z}) \right)^2 - \frac{n_\gamma^2}{64\pi^4} \left(\mathcal{I}_\gamma^{(1)}(\mathbf{z}) \right)^2 \right].$$

(B.4)

The first term is regular in $\mathcal{U}_\pm$, as was noticed in (3.14). Thus, a non-trivial contribution to $\alpha^{[0]}$ originates from the second term only. The latter has two cuts along $\ell_{\pm\gamma}$ and the sum of all integrals reduces to an integral along these contours of the residue of (B.4).[8] As a result, one finds the following contribution

$$-\frac{\mathrm{i}n_\gamma^2}{(4\pi)^4} q_\Lambda p^\Lambda \sum_{s_1, s_2 = \pm} s_1 s_2 \int_0^{\mathrm{i}s_1\infty} \mathcal{D}^{(s_1)}[\mathbf{z}_1] \int_0^{\mathrm{i}s_2\infty} \mathcal{D}^{(s_2)}[\mathbf{z}_2] \frac{2(\mathbf{z}_1 + \mathbf{z}_2)\mathbf{z}}{(\mathbf{z}_1 - \mathbf{z})(\mathbf{z}_2 - \mathbf{z})}$$

$$= -\frac{\mathrm{i}n_\gamma^2}{(4\pi)^4} q_\Lambda p^\Lambda \left(\left(\mathcal{I}_\gamma^{(1)}(\mathbf{z}) \right)^2 + 16 \mathcal{K}_\gamma^2 \right),$$

(B.5)

where we denoted the measure $\mathcal{D}^{(s)}[\mathbf{z}] = \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}} \log(1 - \mathrm{e}^{-2\pi\mathrm{i}s\Xi_\gamma(\mathbf{z})})$. The other terms in $\tilde{T}_\alpha^{[\pm 0]}$ can be treated in the way discussed above. The only subtlety is that reducing the integral along the figure-eight contour for the third term in (B.1) to the integral of its discontinuity, one should take into account the contributions from the poles of the integration measure. After lengthy calculations, one arrives at the total result, which is conveniently formulated in terms of the combination $\tilde{\alpha}$ and coincides with (3.9c). In particular, the contribution quadratic in instantons (B.5) is canceled in this combination and the linear instanton approximation is exact. The only place where the quadratic terms appear is the definition of the field σ in (3.10).

Appendix C. Verification of transition functions

The aim of this appendix is to check that the transition functions (3.39) satisfy all necessary conditions, namely, that they define regular contact twistor lines in the patch $\mathcal{U}_+$ and they are consistent with the transition functions (3.25). But first we need to evaluate their derivatives. As usual, the situation is complicated by the fact that they are written as functions of coordinates in the patch $\mathcal{U}_+$, whereas in the contact transformations they are considered as functions of $\xi_{[+]}^\Lambda$ and $\tilde{\xi}_\Lambda^{[i]}$.

Let us start by evaluating the derivative of $H^{[+i]}$ w.r.t. $\Xi_{\gamma_m}^+$. After integrating by parts in the second and fourth terms, one finds

$$2\mathrm{i}\frac{\partial H^{[+i]}}{\partial \Xi_{\gamma_m}^+} = \frac{n_{\gamma_m}}{4\pi^2} \left(\int_0^{-\mathrm{i}\infty} \frac{\mathcal{D}_{\gamma_m} \Xi}{\Xi_{\gamma_m}^+ - \Xi} + \frac{\pi\mathrm{i}}{12\Xi_{\gamma_m}^+} \right)$$

$$+ \frac{1}{(8\pi^2)^2} \sum_j n_{\gamma_j} q_{j,\Lambda} \frac{\partial \mathcal{S}_j}{\partial \Xi_{\gamma_m}^+} \sum_k n_{\gamma_k} p_k^\Lambda \mathcal{S}_k + \frac{1}{2(8\pi^2)^2} \sum_{j \neq k} n_{\gamma_j} n_{\gamma_k} \langle \gamma_j, \gamma_k \rangle \mathcal{S}_j \frac{\partial \mathcal{E}_k}{\partial \Xi_{\gamma_m}^+}$$

$$- \frac{1}{8\pi^2} \sum_j n_{\gamma_j} \int_0^{-\mathrm{i}\infty} \frac{\mathrm{d}\Xi}{\Xi_{\gamma_j}^+ - \Xi} \left[\left(\frac{\partial \Sigma_{\gamma_j}(\Xi)}{\partial \Xi_{\gamma_m}^+} + \delta_{j,m} \frac{\partial \Sigma_{\gamma_j}(\Xi)}{\partial \Xi} \right) \right.$$

$$- \left. \left(\Sigma_{\gamma_j}(\Xi) - \Sigma_{\gamma_j}(\Xi_{\gamma_j}^+) \right) \left(\frac{\partial}{\partial \Xi_{\gamma_m}^+} + \delta_{j,m} \frac{\partial}{\partial \Xi} \right) \right] \log \left(1 - \mathrm{e}^{-2\pi\mathrm{i}(\Xi - \Sigma_{\gamma_j}(\Xi))} \right).$$

(C.1)

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[8] The additional contribution from the poles $\mathbf{z} = 0, \infty$ of the integration measure in (2.26) is canceled by a similar one from the first term.

Using equation (3.33) for $\Sigma_{\gamma_i}(\Xi)$, it is easy to obtain that the very last term is equal to

$$\frac{1}{32\pi^4} \sum_{j \neq k} n_{\gamma_j} n_{\gamma_k} \langle \gamma_j, \gamma_k \rangle \int_0^{-i\infty} \frac{\mathcal{D}_{\gamma_j} \Xi}{\Xi_{\gamma_j}^+ - \Xi} \int_0^{-i\infty} \frac{\Xi \, d\Xi'}{\Xi_{\gamma_k}^+ \Xi - \Xi_{\gamma_j}^+ \Xi'}$$

$$\times \left(\frac{\partial}{\partial \Xi_{\gamma_m}^+} + \delta_{j,m} \frac{\partial}{\partial \Xi'} \right) \log \left(1 - e^{-2\pi i (\Xi' - \Sigma_{\gamma_k}(\Xi'))} \right). \tag{C.2}$$

On the other hand, the combination of the third and fourth terms gives

$$\frac{1}{32\pi^4} \sum_{j \neq k} n_{\gamma_j} n_{\gamma_k} \langle \gamma_j, \gamma_k \rangle \left\{ \frac{1}{4} \mathcal{E}_j \frac{\partial \mathcal{E}_k}{\partial \Xi_{\gamma_m}^+} + \int_0^{-i\infty} \frac{\mathcal{D}_{\gamma_j} \Xi}{\Xi_{\gamma_j}^+ - \Xi} \int_0^{-i\infty} \frac{d\Xi'}{\Xi_{\gamma_k}^+ \Xi - \Xi_{\gamma_j}^+ \Xi'} \right.$$

$$\times \left. \left[\frac{\delta_{j,m} \Xi' \left(\Xi_{\gamma_j}^+ - \Xi \right) + \delta_{k,m} \Xi^2}{\Xi_{\gamma_k}^+ \Xi - \Xi_{\gamma_j}^+ \Xi'} - \Xi \frac{\partial}{\partial \Xi_{\gamma_m}^+} \right] \log \left(1 - e^{-2\pi i (\Xi' - \Sigma_{\gamma_k}(\Xi'))} \right) \right\}. \tag{C.3}$$

Combining these two contributions and using equation (3.34) for $\mathcal{E}_j$, one arrives at the final result

$$2i \frac{\partial H^{[+i]}}{\partial \Xi_{\gamma_m}^+} = \frac{1}{8\pi^2} \sum_k n_{\gamma_k} \left(\delta_{k,m} + \frac{1}{8\pi^2} \sum_j n_{\gamma_j} q_{j,\Lambda} \frac{\partial \mathcal{S}_j}{\partial \Xi_{\gamma_m}^+} p_k^\Lambda \right) \mathcal{S}_k. \tag{C.4}$$

One immediately concludes that it leads to

$$2i\partial_{\xi_{[+]}^\Lambda} H^{[+i]} = F_\Lambda \left(\xi_{[+]}^\Lambda \right) + \frac{1}{8\pi^2} \sum_j n_{\gamma_j} q_{j,\Lambda} \mathcal{S}_j, \qquad \partial_{\tilde{\xi}_\Lambda^{[i]}} H^{[+i]} = \frac{1}{8\pi^2} \sum_j n_{\gamma_j} p_j^\Lambda \mathcal{S}_j. \tag{C.5}$$

Now it is easy to verify that the contact transformation generated by $H^{[+i]}$ removes all singularities of the contact twistor lines near the north pole. First, the regularity of $\xi_{[+]}^\Lambda, \tilde{\xi}_\Lambda^{[+]}$ trivially follows from the fact that their instanton contributions depend only on the combination $\mathcal{J}_{\gamma_i} + \mathcal{S}_j$, which is regular around $\mathbf{z} = 0$. The analysis of $\alpha^{[+]}$ is as usual a bit more complicated. It can be represented as

$$4i\alpha^{[+]} = \tilde{\alpha}^{[i]} - 2i \left[\tilde{\xi}_\Lambda^{[+]} \xi_{[+]}^\Lambda + \left(2 - \xi_{[+]}^\Lambda \partial_{\xi_{[+]}^\Lambda} - \tilde{\xi}_\Lambda^{[+]} \partial_{\tilde{\xi}_\Lambda^{[i]}} \right) H^{[+i]} - \partial_{\xi_{[+]}^\Lambda} H^{[+i]} \partial_{\tilde{\xi}_\Lambda^{[i]}} H^{[+i]} \right]$$

$$= \tilde{\alpha}^{[i]} - 2i \tilde{\xi}_\Lambda^{[+]} \xi_{[+]}^\Lambda + \frac{i}{4\pi^3} \sum_j n_{\gamma_j} \int_0^{-i\infty} \frac{d\Xi}{\Xi_{\gamma_j}^+ - \Xi} \mathrm{Li}_2 \left(e^{-2\pi i (\Xi - \Sigma_{\gamma_j})} \right)$$

$$+ \frac{1}{8\pi^2} \sum_j n_{\gamma_j} \Xi_{\gamma_j}^+ \mathcal{S}_j - \frac{1}{4\pi^2} \sum_j n_{\gamma_j} \int_0^{-i\infty} \frac{\mathcal{D}_{\gamma_j} \Xi}{\Xi_{\gamma_j}^+ - \Xi} \Sigma_{\gamma_j}(\Xi). \tag{C.6}$$

Note that in the fourth term one can safely replace $\Xi_{\gamma_j}^+$ by Ξ_{γ_j}. Then the discontinuities of this term arising due to the branch cuts of Ξ_{γ_j} cancel those of the last term in (C.6), whereas the discontinuities coming from the cuts of $\mathcal{S}_j$ cancel those of the last two terms in $\tilde{\alpha}^{[i]}$ (3.32). At the same time the third term in (C.6) removes the singularities of the fourth term in (3.32). Since the second term is regular, we conclude that $\alpha^{[+]}$ does not have branch-cut singularities in $\mathcal{U}_+$ except the term coming from the anomalous dimension. Besides, it is easy to show that all simple poles at $\mathbf{z} = 0$ also cancel each other. Thus, the contact twistor line $\alpha^{[+]}$ satisfies all regularity conditions.

Finally, let us check that the proposed form of $H^{[+i]}$ is consistent with the transition functions through the BPS rays (3.25). The latter can be obtained from the composition law (3.20). The term linear in the transition functions gives

$$\frac{i}{2} \left(G_{\gamma_i} - \frac{1}{2} q_{i,\Lambda} p_i^\Lambda \left(G'_{\gamma_i} \right)^2 + \frac{G'_{\gamma_i}}{16\pi^2} \sum_j n_{\gamma_j} \left(q_{i,\Lambda} p_j^\Lambda + q_{j,\Lambda} p_i^\Lambda \right) \mathcal{S}_j + \frac{1}{2} G'_{\gamma_i} \Sigma_{\gamma_i} \left(\Xi_{\gamma_i}^+ \right) \right), \tag{C.7}$$

where $\mathcal{S}_i$ denotes the branch of the function (3.37) obtained by analytical continuation from the patch $\mathcal{U}_i$ clockwise. The quadratic term is evaluated using (C.5) and leads to

$$-\frac{\mathrm{i}}{16\pi^2} q_{i,\Lambda} G'_{\gamma_i} \sum_j n_{\gamma_j} p_j^\Lambda \mathcal{S}_j. \tag{C.8}$$

Altogether these contributions reproduce our initial starting point (3.25). This completes the verification that (3.25) and (3.39) form a consistent set of transition functions.

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